If and only if

If and only if statements

A theorem that asserts that two statements P and Q are equivalent requires you to prove *both* $P \implies Q$ and $Q \implies P$.

$(P \Leftrightarrow Q)$ is equivalent to

$P \Rightarrow Q$ and $Q \Rightarrow P$.

Chapter 7 exercise 3.

Proposition: If a is an integer, then a is even if and only if $a^3 + a^2 + a$ is even.

There are two claims:

a even implies $a^3 + a^2 + a$ is even.

$a^3 + a^2 + a$ is even implies a is even.

Each requires proof.

Proof: First we show that, if a is even, then $a^3 + a^2 + a$ is even.

if a is even, then $a = 2K$ for some integer K.

Therefore $a^3 + a^2 + a = (2K)^3 + (2K)^2 + 2K$

$$= 8K^3 + 4K^2 + 2K = 2(4K^3 + 2K^2 + K)$$

Since we see that $a^3 + a^2 + a = 2m$ where

$m = 4K^3 + 2K^2 + K$,

we conclude that $a^3 + a^2 + a$ is even.

Now we show that, if $a^3 + a^2 + a$ is even, then a is even.

a^3+a^2+a even $\Rightarrow$ a even

is the same as

a odd $\Rightarrow$ a^3+a^2+a odd.

a odd means $a = 2k+1$ for some $k \in \mathbb{Z}$.

$a^3+a^2+a = (2k+1)^3+(2k+1)^2+(2k+1)$

a^3+a^2+a is a sum of 3 odd numbers

a^3+a^2 is even (sum of 2)

$(a^3+a^2)+a$ is odd (odd + even)

Conclude that a^3+a^2+a is odd.

We've shown a^3+a^2+a is even $\Leftrightarrow$ a is even.

Chapter 7, exercise 9.

Proposition: Suppose that a is an integer. Then $14|a$ if and only if $7|a$ and $2|a$.

Proof: First we suppose that $14|a$ and show that both 7 and 2 divide a.

Suppose $14 | a$. Then $a = 14k$ for some $k \in \mathbb{Z}$.

Therefore $a = 7(2k)$ ~~fa~~

so $7 | a$

and $a = 2(7k)$

so a is a multiple of 2.

Now we show that, if both 7 and 2 divide a, then 14 divides a.

Side remark: Suppose 6 and 4 divide a. Does $24 | a$?

No: consider 12. $6 | 12$ and $4 | 12$, but $24 | 12$.

We know by unique factorization into primes that a is a product of prime numbers and the list of such primes is determined by a. Also, 2 and 7 are prime numbers. Since $2 | a$, 2 must be one of the primes dividing a. Since $7 | a$, 7 must be one of those primes. So the prime factorization of $a = 2 \cdot 7 \cdot$ (possibly other primes).

Therefore $14 | a$.

Equivalence

with real number entries,

Theorem Suppose A is an $n \times n$ matrix. The following statements are equivalent:

(a) The matrix A is invertible.

(b) The equation $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b} \in \mathbb{R}^n$.

(c) The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(d) The reduced row echelon form of A is I_n.

(e) $\det(A) \neq 0$.

(f) The matrix A does not have 0 as an eigenvalue.

A invertible

$Ax = b \Rightarrow$

$x = A^{-1}b$

$Ax = 0$

Figure 1: Theorem from page 149

Cycle proofs

If each step in the circle of implications:

$P_1 \Rightarrow P_2 \quad P_2 \Rightarrow P_3 \quad P_3 \Rightarrow P_4 \cdots \quad P_n \Rightarrow P_1$

$$P_1 \implies P_2 \implies P_3 \implies \cdots \implies P_n \implies P_1$$

is true, then all of the statements are equivalent – that is, all true or all false together.

Suppose P_1 is true $P_1 \Rightarrow P_2$ true means P_2 true

P_2 true so P_3 true .... P_n is true therefore P_1 is true

If P_1 is false, $P_n \Rightarrow P_1$ is true so P_n is false

P_{n-1} is false .... all statements are false.