Congruence arithmetic.

$n \in \mathbb{N}$ $x, y \in \mathbb{Z}$

$n | x$ and $n | y$ then $n | (x+y)$.

Lemma: If x and y are multiples of n, so is $x + y$.

Proof: $x = kn$ for some k, $y = jn$ for some j so $x+y = kn + jn = (k+j)n$.

Proposition: Given integers a, b, c, d and a natural number n,
if $a \equiv b \pmod{n}$ then

- If $c \equiv d \pmod{n}$, then $a + c \equiv b + d \pmod{n}$

$$\begin{array}{c} a \equiv b \pmod{n} \\ c \equiv d \pmod{n} \\ \hline a + c \equiv b + d \pmod{n} \end{array}$$

Proof: We know $a-b$ is a multiple of n

we know $c-d$ is a multiple of n

$(a-b)+(c-d)$ is a multiple of n

$(a+c)-(b+d)$ is a multiple of n

$a+c \equiv b+d \mod n$.

- $ac \equiv bc \pmod{n}$

$a \equiv b \pmod{n}$

$ca \equiv cb \pmod{n}$

Proof: $ac - bc = c(a-b)$

$a - b = kn$ for some k.

$ac - bc = c \cdot k \cdot n$ which is a multiple of n. so $ac \equiv bc \mod n$.

Congruence arithmetic continued

Proposition:

$a^r \equiv b^r \pmod{n}$ for any natural number r.

Lemma from algebra.

$$x^2 - y^2 = (x-y)(x+y)$$

$$x^3 - y^3 = (x-y)(x^2 + xy + y^2)$$

$$x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots + y^{n-1})$$

Proof of Proposition.

$$a^r - b^r = (a-b)(a^{r-1} + \cdots + b^{r-1})$$

$a \equiv b \bmod n$, we know

$$a - b = kn.$$

$$a^r - b^r = kn(a^{r-1} + \cdots + b^{r-1})$$

so $a^r - b^r$ is a multiple of n.

$$\begin{array}{rl}
 & x^{n-1} + x^{n-2}y + \cdots + y^{n-1} \\
 & \qquad\qquad x - y \\
\hline
 & x^n + x^{n-1}y + x^{n-2}y^2 + \cdots + xy^{n-1} \\
 & \quad - x^{n-1}y - x^{n-2}y^2 - \cdots - xy^{n-1} - y^n \\
\hline
 & x^n - y^n
\end{array}$$

Casting out nines

387

$3+8+7 = 18$

$1+8 = 9$

345

$3+4+5 = 7+5 = 12$

$1+2 = 3$

$387 = 9 \cdot 43$

$387 = 3 \cdot 100 + 8 \cdot 10 + 7 \cdot 1$

$= \underline{3 \cdot 10^2} + 8 \cdot 10 + 7 \cdot 1$

$10 \equiv 1 \mod 9$

$10^2 \equiv 1 \mod 9$

$7 \cdot 1 \equiv 7 \mod 9$

$8 \cdot 10 \equiv 8 \mod 9$

$3 \cdot 10^2 \equiv 3 \mod 9$

$3 \cdot 10^2 + 8 \cdot 10 + 7 \cdot 1 \equiv \underbrace{3+8+7}_{11} \mod 9$

$8 + 1 \cdot 10 \equiv 8 + 1 \quad 18 \mod 9$

$\equiv 9 \equiv 0 \mod 9$

$$345 = 3\cdot 10^2 + 4\cdot 10 + 5\cdot 1$$

$$10^2 \equiv 1 \quad 10 \equiv 1 \quad 1 \equiv 1 \mod 9$$

$$345 = 3+4+5 \mod 9$$

$$\equiv 3 \mod 9.$$

$\cancel{11 \equiv -1 \text{ mod}}$

$10 \equiv -1 \mod 11.$

345

$3 - 4 + 5 = 4 \not\equiv 0 \mod 11$

$11 \nmid 345$