Proof by Contrapositive

The contrapositive.

Important: The contrapositive of an implication $P \implies Q$ is $\sim Q \implies \sim P$.

- If $\sim Q$ is false (meaning Q is true) the implication $\sim Q \implies \sim P$ is automatically true.
- So we assume $\sim Q$ is true – that is, that Q is false – and try to conclude that $\sim P$ is true – meaning that P is false.

$$P \Rightarrow P_1 \Rightarrow P_2 \Rightarrow \dots \Rightarrow Q$$

$$\sim Q \Rightarrow Q_1 \Rightarrow Q_2 \Rightarrow \dots \Rightarrow \sim P$$

P	Q	$P \Rightarrow Q$	$\sim Q$	$\sim P$	$\sim Q \Rightarrow \sim P$	$Q \Rightarrow P$
T	T	T	F	F	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	F
F	F	T	T	T	T	T

NOT THE SAME!!

Contrapositive.

Proposition: Suppose that $\underline{x \in \mathbb{Z}}$. Suppose $\overbrace{\underline{x^2 - 4x + 3 \text{ is even}}}^{P}$. $\underbrace{\text{Then } x \text{ is odd}}_{Q}$.

$a = x^2 - 4x + 3$

$a = 2m = x^2 - 4x + 3 = (x-1)(x-3)$ HINT FOR DIRECT PROOF

$\sim Q$: x is NOT odd so x is even.

$\sim P$ $x^2 - 4x + 3$ is NOT even so $x^2 - 4x + 3$ is odd.

Suppose x is even. Then $x^2 - 4x + 3$ is odd.

$x = 2m$ $\qquad (2m)^2 - 4(2m) + 3 \leftarrow$ odd

Contrapositive

Proof: Suppose x is even. Then $x = 2m$ for some integer m. Therefore

$$B = x^2 - 4x + 3 = 4m^2 - 8\cancel{x}m + 3 = 2(2m^2 - \cancel{2}4m + 1) + 1.$$

Since B is of the form $2k + 1$ with $k = 2m^2 - \cancel{2}4m + 1$, we conclude that B is odd. Therefore B is not even. We have shown that if x is not odd, then B is not even, and therefore if B is even, x is odd.

Contrapositive

$a=4, b=3.$

Proposition: [Suppose that $x \cancel{\in \mathbb{Z}}$, ~~that~~ that a is even, and that b is odd.] $\boxed{\text{If } x^2 - ax + b \text{ is even, then } x \text{ is odd.}}$

Assume x is even. Then $x = 2m$ for some $m \in \mathbb{Z}$.

Therefore

$$x^2 - ax + b = (2m)^2 - 2am + b$$

$(2m)^2 - 2am = 4m^2 - 2am = 2(2m^2 - am)$ is even.

b is odd.

$(2m)^2 - 2am + b$ is an even integer plus an odd integer, so it's odd.

$$\underbrace{\forall x \in \mathbb{Z}, \forall a \text{ even in } \mathbb{Z}, \forall b \text{ odd in } \mathbb{Z},}_{R} \quad x^2 - ax + b \text{ even} \Rightarrow x \text{ odd}.$$

$$R \Rightarrow (P \Rightarrow Q)$$

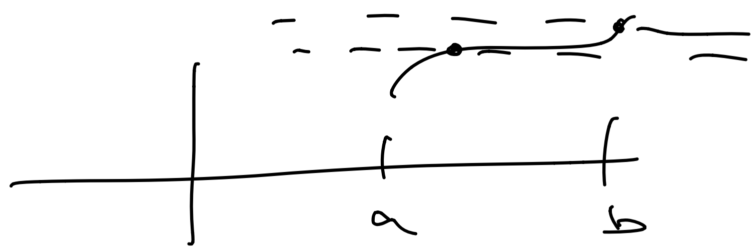
$$R \Rightarrow (\sim Q \Rightarrow \sim P)$$

An example from calculus

Theorem: Let $f : [a, b] \to \mathbb{R}$ be a function that is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). If $f'(x) = 0$ for all $x \in [a, b]$, then f is constant.

Proof: NOT (For all $x \in (a,b)$, $f'(x) = 0$) $\iff$ (there exists x, $f'(x) \neq 0$)

- We will show that if f is not constant, then there is an $x \in [a, b]$ with $f'(x) \neq 0$.
- Suppose that $f(x)$ is not constant. Then there are two (different) points u and v in $[a, b]$ such that $f(u) \neq f(v)$.

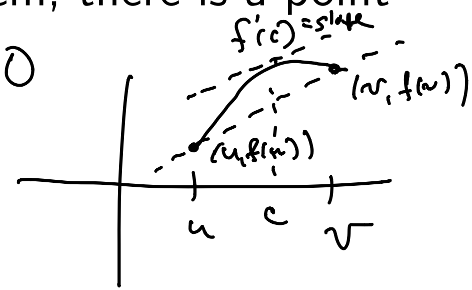


calculus cont'd

- $f : [u, v] \to \mathbb{R}$ is continuous on $[u, v]$ and differentiable on (u, v). Therefore, by the mean value theorem, there is a point $c \in (u, v)$ such that

$$f'(c) = \frac{f(v) - f(u)}{v - u}.$$

$\neq 0$



Since $f(v) \neq f(u)$, the quantity on the right is not zero, and so $f'(c) \neq 0$.

- Therefore $f'(x)$ is not zero for all $x \in [a, b]$. This proves our result.