Direct Proof

Direct Proof

From page 122 of the text

by showing, first, that $m \leq n$, and then that $m \geq n$.

Proposition If $a, b, c \in \mathbb{N}$, then $\text{lcm}(ca, cb) = c \cdot \text{lcm}(a, b)$.

Proof. Assume $a, b, c \in \mathbb{N}$. Let $m = \text{lcm}(ca, cb)$ and $n = c \cdot \text{lcm}(a, b)$. We will show $m = n$. By definition, $\text{lcm}(a, b)$ is a positive multiple of both a and b, so $\text{lcm}(a, b) = ax = by$ for some $x, y \in \mathbb{N}$. From this we see that $n = c \cdot \text{lcm}(a, b) = cax = cby$ is a positive multiple of both ca and cb. But $m = \text{lcm}(ca, cb)$ is the *smallest* positive multiple of both ca and cb. Thus $m \leq n$.

First we prove $m \leq n$

On the other hand, as $m = \text{lcm}(ca, cb)$ is a multiple of both ca and cb, we have $m = cax = cby$ for some $x, y \in \mathbb{Z}$. Then $\frac{1}{c}m = ax = by$ is a multiple of both a and b. Therefore $\text{lcm}(a, b) \leq \frac{1}{c}m$, so $c \cdot \text{lcm}(a, b) \leq m$, that is, $n \leq m$.

Then we prove $n \leq m$

We've shown $m \leq n$ and $n \leq m$, so $m = n$. The proof is complete. ■

Figure 1: lcm proposition

The hidden part - check the definitions

Definition: Let a and b be positive integers. Then the least common multiple $\mathrm{lcm}(a, b)$ is the smallest positive integer m such that $a|m$ and $b|m$.

$a = 2, b = 3$ $\mathrm{lcm}(2,3)$ is smallest positive integer m such that $2|m$ and $3|m$.

$m = 2x$ $m = 3y$

6 is a multiple of 2 and 3.

6

2, 4, 6, ...

3, 6, ...

$\mathrm{lcm}(a,b) \leq ab$

sometimes =

sometimes NOT

$a = 6, b = 8$

48 is a common multiple.

24 is also a common multiple.

8, 16, 24

6, 12, 18, 24

The hidden part II

Second, make sure the claim is clear. Look at some examples.

$\mathrm{lcm}(ca, cb) = c \cdot \mathrm{lcm}(a,b)$.

$\mathrm{lcm}(2,3) = 6$. $\quad c = 5$

Prop says $\mathrm{lcm}(5\cdot 2, 5\cdot 3) = \mathrm{lcm}(10, 15) = 5 \cdot \mathrm{lcm}(2,3) = 5\cdot 6 = \underline{30}$

10, 20, 30, ...

15, 30, ...

$\mathrm{lcm}(6,8) = 24$ $\quad c = 2$. $\quad \mathrm{lcm}(12,16) = 2\cdot \mathrm{lcm}(6,8) =$

$2 \cdot 24 = \cancel{48}\ \underline{48}$

12, 24, 36, 48, . . .

16, 32, 48, . . .

The hidden part continued III - interpret the definition

Three ways of saying the same thing:

- $\cancel{x}$ m is the smallest positive integer such that $a|m$ and $b|m$
- If x is a positive integer so that $a|x$ and $b|x$, then $x \geq \mathrm{lcm}(a, b)$.
- If x is a positive integer so that $a|x$ and $b|x$, then $\mathrm{lcm}(a, b) \leq x$.

x is smallest element of a set A

$\Updownarrow$

$x \in A$ and $\forall a \in A$, $x \leq a$.

The hidden part IV

Read the proof to understand it's structure, without worrying about the details.

Take the proof of the proposition apart

- Assume $a, b, c \in \mathbb{N}$. (a, b, c, are all positive)
- Let $m = \mathrm{lcm}(ca, cb)$ and $n = c\mathrm{lcm}(a, b)$. We will show that $m = n$. ($m \geq n$ for, then $m \leq n \implies m = n$)
- By definition, $\mathrm{lcm}(a, b)$ is a positive multiple of both a and b, so $\mathrm{lcm}(a, b) = ax = by$ for some x and y in $\mathbb{N}$.
- From this we see that $n = c\mathrm{lcm}(a, b) = cax = cby$ is a positive multiple of both ca and cb. Thus $m \leq n$.

n is a common multiple of ca, cb and m is least common multiple so $n \geq m$.

$c\,\mathrm{lcm}(a,b) = cax$

$c\,\mathrm{lcm}(a,b) = cby$

Taking the proof apart

- On the other hand, as $m = \operatorname{lcm}(ca, cb)$ is a multiple of both ca and cb, we have $m = cax = cby$ for some $x, y \in \mathbb{Z}$.
- Then $\frac{1}{c}m = ax = by$ is a multiple of both a and b.

  $c \neq 0$?
  $c \in \mathbb{N}$. ✓
- Therefore $\operatorname{lcm}(a, b) \leq \frac{1}{c}m$ so $c\operatorname{lcm}(a, b) \leq m$, that is $n \leq m$.
- Since $m \leq n$ and $n \leq m$, we have $m = n$. The proof is complete.