Definitions

An example from linear algebra

Definition: A set $\{v_1 \ldots, v_n\}$ of elements of a vector space V is called *linearly independent* if, for any set $\{a_1, \ldots, a_n\}$ of scalars, if $\sum_{i=1}^{n} a_i v_i = 0$ then $a_i = 0$ for all $i = 1, \ldots, n$. A set of vectors that is not linearly independent is called *linearly dependent.*

A set $\{v_1, \ldots, v_n\}$ is linearly independent $\iff$ For any set $\{a_1, \ldots, a_n\}$ of scalars such that $\sum_{i=1}^{n} a_i v_i = 0$, we have $a_i = 0$, $i = 1, \ldots, n$.

P

If $\sum a_i v_i = 0$ Then all $a_i = 0$.

Q R

$\left[P \iff (Q \Rightarrow R) \right]$ true

Linearly Independent

NOT " "

$Q \Rightarrow R$

False that $Q \Rightarrow R$

Theorem: The set of vectors $\{(1,3),(2,2)\}$ is linearly independent in $\mathbb{R}^2$.

Proof: We must show that

$$\underbrace{a_1(1,3)+a_2(2,2)=0}_{P} \implies \underbrace{a_1=0 \text{ and } a_2=0}_{Q}$$

$(\sum a_i v_i)$

Assume $a_1(1,3)+a_2(2,2)=0$

Therefore $(a_1, 3a_1) + (2a_2, 2a_2) = 0$

$$(a_1+2a_2, 3a_1+2a_2) = 0$$

$$a_1 + 2a_2 = 0$$

$$3a_1 + 2a_2 = 0.$$

$2a_1 = 0$ so $a_1 = 0$

Then $2a_2 = 0$ so $a_2 = 0$

so $a_1 = a_2 = 0.$

Therefore the set is linearly independent.

Theorem: The set of vectors $\{(1,1,1),(2,2,2),(1,3,2)\}$ is linearly dependent in $\mathbb{R}^3$.

Proof: We must show that

$$\underbrace{\sum a_i v_i = 0}_{P} \Rightarrow \underbrace{\text{all } a_i = 0}_{Q} \quad \text{is FALSE}$$

P	Q	P ⇒ Q
T	F	F
⋮	⋮	T

$P \Rightarrow Q$ is FALSE if ~~P~~ and only if P is True ✓ Q is FALSE

$\sum a_i v = 0$ and not all $a_i = 0$.

$a_1(1,1,1) + a_2(2,2,2) + a_3(1,3,2) = 0$.

$(a_1 + 2a_2 + a_3,\ a_1 + 2a_2 + 3a_3,\ a_1 + 2a_2 + 2a_3) = 0$.

$a_1 = 2 \quad a_2 = -1 \quad a_3 = 0$.

Then $a_1(1,1,1) + a_2(2,2,2) + a_3(1,3,2) = 0$.

We've shown that the vectors are linearly dependent.