Direct Proofs Example

Pg 122 - 123

The arithmetic/geometric mean inequality

p 122 - 123

Definition: The arithmetic mean of two real numbers a and b is $(a+b)/2$.

Definition: The geometric mean of two positive real numbers a and b is $\sqrt{(}ab)$.

Proposition: If a and b are positive real numbers, then the geometric mean of a and b is less than or equal to their arithmetic mean.

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$$\underbrace{\sqrt{(ab)}}_{\text{geometric mean}} \leq \underbrace{\frac{(a+b)}{2}}_{\text{arithmetic mean}}$$

Problem Solving Phase

Show: $\sqrt{ab} \leq \frac{(a+b)}{2}$?? ?? Lemma: $a \geq b$ then $c^2 \geq b^2$??

$ab \leq \left(\frac{a+b}{2}\right)^2$

Need to know:

$4ab \leq (a+b)^2 = a^2+2ab+b^2$

if $a \geq b$, $a \geq 0$, $b \geq 0$

is $\sqrt{a} \geq \sqrt{b}$?

$0 \leq a^2-2ab+b^2 = (a-b)^2$

Isolating the needed lemma

Lemma: If a and b are positive, and $a \leq b$, then $\sqrt{a} \leq \sqrt{b}$ where $\sqrt{x}$ denotes the positive square root of x.

$u^2 - v^2 = (u-v)(u+v)$

Given $b - a \geqslant 0$ Want $\sqrt{b} - \sqrt{a} \geqslant 0$

Know $\sqrt{a} + \sqrt{b} \geqslant 0$

$(\sqrt{a}+\sqrt{b})(\sqrt{b}-\sqrt{a}) = b - a = (\sqrt{b})^2 - (\sqrt{a})^2$

(HIDDEN) $= b - a$

$(\sqrt{a}+\sqrt{b})(\sqrt{b}-\sqrt{a}) = b - a \geqslant 0$

$\sqrt{a} + \sqrt{b} \geqslant 0$

Given $b \geqslant a$. Therefore $b - a \geqslant 0$. Therefore $b - a = (\sqrt{b}+\sqrt{a})(\sqrt{b}-\sqrt{a}) \geqslant 0$. Since $\sqrt{b}+\sqrt{a} \geqslant 0$, we can divide both sides of this inequality by $\sqrt{b}+\sqrt{a}$ to obtain $\sqrt{b}-\sqrt{a} \geqslant 0$ so $\sqrt{b} \geqslant \sqrt{a}$.

The proof

Proposition: If a and b are positive real numbers, then:

$$\sqrt{(ab)} \leq \frac{(a+b)}{2}$$

Proof:

We know that $(a-b)^2 \geq 0$. Therefore $a^2 + b^2 - 2ab \geq 0$ and so $a^2 + b^2 \geq 2ab$. Add $2ab$ to both sides to obtain $a^2 + 2ab + b^2 \geq 4ab$ so $(a+b)^2 \geq 4ab$. Both sides of this inequality are positive, since the left side is a square the right side is a product of positive numbers. Now apply the lemma to take the square root of both sides to obtain

$$(a+b) \geq 2\sqrt{ab}.$$

Dividing both sides by 2 yields the desired result.