Pascal's Triangle and the binomial theorem

Pascal's Triangle

We proved that $\binom{n}{k}$ counts the number of subsets with k elements that can be found in a set with n elements.

We know that

$$\to \binom{n}{k} = \cancel{\frac{n!}{(n-k)!k!}} \quad \frac{n!}{(n-k)!\,k!} \quad 0 \le k \le n$$

$\binom{n}{k} = 0$ if $k > n$.

We set $\binom{n}{k} = 0$ if $k > n$ (there are no subsets with k elements in a set with n elements if $k > n$.)

In the inductive proof that $\binom{n}{k}$ counts subsets, we proved that

$$\boxed{\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}}$$

$\binom{n}{k} = 0$ if $k < 0$

This relation defines "Pascal's Triangle".

1 $\binom{0}{0}$

1 1 0 $\binom{1}{0}$ $\binom{1}{1}$

$n=2$ $\binom{2}{0}$ 1 2 1

1 3 $\binom{3}{1}$ 3 $\binom{3}{2}$ 1

1 4 6 $\binom{4}{2}$ 4 $\binom{4}{3}$ 1

1 5 10 $10 = \binom{5}{3}$ 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

⋮

Binomial Theorem

Theorem:

$$\underline{(x+y)^n} = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$$

Proof: By induction.

$(x+y)^2 = x^2 + 2xy + y^2$

$(x+y)^3 = x^3 + \underline{3}x^2y + \underline{3}xy^2 + y^3$

$\underline{\text{Proof}}$: $n=1$ $(x+y)^1 = x+y = \binom{1}{0}x + \binom{1}{1}y$

Suppose $(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$

Compute $(x+y)^{n+1} = (x+y)\sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$

$= \sum_{i=0}^{n} \binom{n}{i} x^{i+1} y^{n-i} + \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i+1}$.

$\underline{xy^n} \quad x^2y^{n-1} \quad x^3y^{n-2} \cdots x^{n+1} \;\Big|\; y^{n+1} \quad \underline{xy^n} \quad x^2y^{n-1} \cdots x^ny$

coeff of x^{n+1} $\binom{n}{n} = 1$ coeff of y^{n+1} $\binom{n}{0} = 1$

coeff of $x^i y^{n+1-i}$ $\binom{n}{i-1}$ on left

coeff of $x^i y^{n+1-i}$ $\binom{n}{i}$ on right

total coeff of $x^i y^{n+1-i} = \binom{n}{i} + \binom{n}{i-1}$

$\sum_{i=0}^{n+1} \underbrace{\left[\binom{n}{i} + \binom{n}{i-1}\right]}_{\binom{n+1}{i}} x^i y^{n+1-i} = \sum_{i=0}^{n+1} \binom{n+1}{i} x^i y^{n+1-i}$

which is what the binomial theorem asserts.

$(x+y)^4$?

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + 3y^3$$

$$(x+y)(x+y)^3 = x^4 + 3x^3y + 3x^2y^2 + xy^3$$

$$x^3y + 3x^2y^2 + 3xy^3 + y^4$$

$$x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

1	3	3	~~1~~		
	1	3	3	1	
1	4	6	4	1	
	1	4	6	4	1
1	5	10	10	5	1

$\vdots$