Combinations

Counting Subsets

Our next problem is counting subsets of a given size chosen from a set of a given size.

Question: How many different k element subsets does a set with n elements have?

Set $X = \{A, B, C, D\}$ $n = 4$ $K = 2$

SUBSETS

$\{A, B\}$ $\{B, C\}$ $\{C, D\}$ total of 6

$\{A, C\}$ $\{B, D\}$

$\{A, D\}$

permutations

AB AD BD 12

BA DA DB $4 \cdot 3 = P(4,2)$

AC BC CD $= \frac{4!}{2!}$

CA CB DC

$n = 4,\ K = 3$

$\{A, B, C\}$ $\{A, C, D\}$

$\{A, B, D\}$ 4 subsets with 3 elts

$\{B, C, D\}$

1 elt: $\{A\}, \{B\}, \{C\}, \{D\}$ 4

4 elt $\{A, B, C, D\}$ 1

0 elt: ϕ

Total of 16 subsets $= 2^4$

#elts	0	1	2	3	4
	1	4	6	4	1

Theorem on counting subsets

Proposition: The number of k element subsets of a set with n elements is called $\binom{n}{k}$. This number is read "n choose k" and it is called a "binomial coefficient". The formula for $\binom{n}{k}$ is:

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

Proof: First we give the book's proof.

$n=4 \quad K=2 \qquad P(n,2) = 4\cdot 3 = 12$

$X = \{A, B, C, D\}$

$\{A,B\} \qquad \{C,D\}$

AB AC AD BC BD CD

BA CA DA CB DB DC

$$\underbrace{\#\text{permutations of 2 elements}}_{P(n,2)} = 2 \cdot \underline{\#\text{ of subsets with 2 elements}}$$

$$\frac{n!}{(n-2)!\,2!} = \frac{P(n,2)}{2} = \binom{n}{2}$$

$\binom{n}{K}$

$n=4$

$\binom{n}{0} = \frac{4!}{4!\cdot 0!} = 1,$

$\binom{n}{1} = \frac{4!}{3!\cdot 1!} = 4,$

$\binom{n}{2} = \frac{4!}{2!\cdot 2!} = \frac{4!}{4} = 6$

$\binom{n}{3} = \frac{4!}{3!\cdot 1!} = 4.$

$\binom{n}{4} = \frac{4!}{1!\,4!} = 1.$

Count k element subsets of a set with n elements.

S_1 with k elements $\qquad S_2$ with k elements $\quad \ldots\ldots \quad S_?$ with k elements

$K!$ permutations of the elts of S_1 $\qquad K!$ permutations of elts of S_2 $\qquad K!$ permutations

grid contains every k permutation

$\#$ of elts in grid is $P(n,K) = \frac{n!}{(n-K)!}$

$\#$ of rows $= K!$.

$$(K!)(\#\text{ of subsets with } k \text{ elements}) = \frac{n!}{(n-K)!}$$

$$\binom{n}{K} = \#\text{ of subsets with } k \text{ elements} = \frac{n!}{(n-K)!\,K!}$$

$\binom{n}{k}$ = # of k element subsets of a set with n elements. $\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$

Proof: now we give a proof by strong induction.

First consider $n=1$. Our set has 1 element.

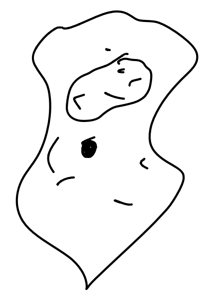
$\binom{1}{0}$ $\binom{1}{1}$

$\binom{1}{0}=1$ $\emptyset$ is only 0-element subset

$\binom{1}{1}=1$ set itself is only 1-element subset. $0!$

We assume now that if X has $n-1$ (or fewer) elements and $0 \le k \le n-1$ then # of subsets with k elements from our set X with $n-1$ elements is

$$\binom{n-1}{k} = \frac{(n-1)!}{k!\,(n-1-k)!}$$

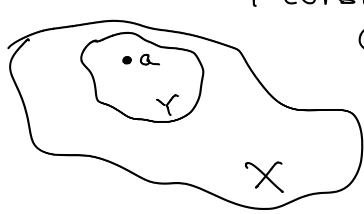


Suppose X has n elements.
Pick an element $a \in X$
Split the k-element subsets of X into two classes:

Class 1: $a \in Y$, $Y \subseteq X$, $|Y| = k$.

Class 2: $a \notin Y$, $Y \subseteq X$, $|Y| = k$

Class 1 Y consists of a and $k-1$ other elements of X. Those are chosen from $X - \{a\}$.



$Y = \{a\} \cup Y_1$

Y_1 is a subset of $X - \{a\}$ with $k-1$ elements.

$\# Y_1 = \binom{n-1}{k-1}$

Class 2 Y consists of k elements chosen from $X - \{a\}$.

# of possible Y = $\binom{n-1}{k}$

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

$$\binom{n-1}{k-1} = \frac{(n-1)!}{[n-1-(k-1)]!\,(k-1)!} \qquad \binom{n-1}{k} = \frac{(n-1)!}{(n-1-k)!\,k!}$$

$$= \frac{(n-1)!}{(n-k)!\,(k-1)!} \qquad\qquad = \frac{(n-1)!}{(n-1-k)!\,k!}$$

$$\binom{n}{k} = \frac{(n-1)!\cdot k}{(n-k)!\cdot k!} + \frac{(n-k)(n-1)!}{(n-k)!\,k!}$$

$$= \frac{\overset{(n-1)!}{(n-1)!}\cdot\overset{n}{(k+n-k)}}{(n-k)!\,k!} = \frac{n!}{(n-k)!\,k!}$$

Examples

Problem 2, page 89: If a set A has 100 elements, how many subsets of A have 5 elements? How many have 10 elements? How many have 99 elements?

\# with 5 elements:

$$\binom{100}{5} = \frac{100!}{5!\cdot 95!} = \frac{100\cdot \cancel{99}\cdot \cancel{98}\cdot 97\cdot 96}{1\cdot \cancel{2}\cdot \cancel{3}\cdot 4\cdot \cancel{5}}$$

$$= 10\cdot 33\cdot 98\cdot 97\cdot 24$$

$$= 10\cdot 33\cdot 98\cdot 97\cdot 24$$

$$\binom{100}{10} = \frac{100!}{10!\,90!} = \frac{100\cdot 99\cdot 98\cdot 97\cdot 96\cdot 95\cdot 94\cdot 93\cdot 92\cdot 91}{10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}$$

$$\binom{100}{99} = \frac{100}{99!\cdot 1!} = 100$$

Problem 5, page 89: How many 16 digit binary strings contain exactly seven 1's?

_ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _

Pick 7 of 16 slots to put a 1

$\binom{16}{7} = \frac{16!}{9! \cdot 7!}$ possibilities.

Problem 11, page 89: How many positive 10 digit integers contain no zeros and exactly three 6's?

1, 2, 3, - - - , 9

10 spots with 9 choices for each spot 10^9 integers (ignoring "six" condition)

_ _ _ _ _ _ _ _ _ _

$\binom{10}{3}$ ways to put 3 6's in these 10 slots.

7 slots left. Put 1, 2, 3, 4, 5, 7, 8, 9

7^8 ways to fill remaing slots.

Total: $\binom{10}{3}7^8$

Problem 19, page 89: A 5-card poker hand is called a *flush* if all cards are the same suit. How many different flushes are there?

cards:

13 hearts $\longrightarrow$ pick 5 $\binom{13}{5}$

13 spades $\longrightarrow$ pick 5 $\binom{13}{5}$

13 clubs $\longrightarrow$ pick 5 $\binom{13}{5}$

13 diamonds $\longrightarrow$ pick 5 $\binom{13}{5}$

total $= 4\binom{13}{5}$