

Permutations

$$4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$$

Factorials

$$0! = 1$$

$$1! = 1 \quad 2! = 2 \cdot 1 = 2$$

$$3! = 1 \cdot 2 \cdot 3 = 6$$

Definition 1: For $n \in \mathbb{Z}$, $n \geq 0$, define $0! = 1$ and $n! = (1)(2) \cdots (n-1)(n)$.
Alternatively, define $n!$ for non-negative integers n by setting $0! = 1$ and $n! = n(n-1)!$.

Proposition: The number of different lists of length n made up of elements from the set $\{1, 2, \dots, n\}$, without repetitions, is $n!$.

$$\{1, 2\} \leftarrow \begin{matrix} (1, 2) \\ (2, 1) \end{matrix} \quad 2 \text{ lists}$$

$$X = \emptyset \quad () \quad 1$$

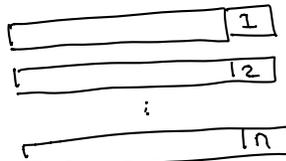
$$\{1, 2, 3\} \leftarrow \begin{matrix} (1, 2, 3) \\ (1, 3, 2) \\ (2, 1, 3) \\ (2, 3, 1) \\ (3, 1, 2) \\ (3, 2, 1) \end{matrix} \quad 6 \text{ lists}$$

Count:
 n choices for first position
 $n-1$ second
 $n-2$ third
 \vdots
 1 choice

= $n!$
is product

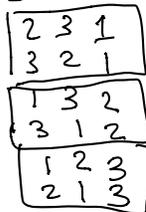
Proof: Case $n=1$. $X = \{1\}$ There is only 1 list (1),
 must show:
 if # of lists of length $n-1$ is $(n-1)!$ \Rightarrow # of lists of length n is $n!$

$X = \{1, \dots, n\}$
 List of length $n \leftrightarrow$ list of length $n-1$ + final element.
 Lists of length n divide up into n groups; S_1, \dots, S_n
 $S_i = \{\text{lists with } i \text{ in last position}\}$



of lists = $|S_1| + |S_2| + \dots + |S_n|$
 $|S_i|$ consists of a list of length $n-1$ using numbers $2, \dots, n$.
 So $|S_i|$ has $(n-1)!$ elements
 Similarly each $|S_i|$ has $(n-1)!$ elements
 Total # = $|S_1| + |S_2| + \dots + |S_n| = n \cdot (n-1)! = n!$
 That's what we wanted to show.

$$X = \{1, 2, 3\}$$



Permutations

Definition: Let X be a set. A permutation of X is a list of length $|X|$ of the elements of X , without repetition. (**Note:** There are other definitions of permutations in other contexts, all related to this one).

Examples:

Permutation of $\{2, 3, 4\}$ is a list $(2, 3, 4)$
there are $6 = 3!$ such permutations

$X = \{A, B, C, D\}$
permutations of X :

ABCD
BACD
CABD
⋮

} 24 permutations
24 = 4!

3, 2, 4
2, 4, 3
4, 2, 3
4, 3, 2
3, 4, 2

Definition: Let X be a set. A k -permutation of X is a list of k elements of X without repetition. $P(n, k)$ is the number of k permutations of a set with n elements.

Proposition: The number $P(n, k)$ of k -permutations of a set with n elements is $n(n-1)\dots(n-k+1)$ or, equivalently

k terms in this product

$$P(n, k) = \frac{n!}{(n-k)!}$$

Proof:

$$\frac{n!}{(n-k)!} = \frac{1 \cdot 2 \cdot \dots \cdot n}{1 \cdot 2 \cdot \dots \cdot (n-k)} = (n-k+1) \cdot \dots \cdot n$$

Multiplication principle.

n choices for 1st slot
 $n-1$ choices for 2nd slot
 \vdots

$n-k+1$ choices for k^{th} slot

$$n(n-1)\dots(n-k+1)$$

Example: $X = \{A, B, C, D\}$

$n = 4$

$k = 2$

- | | |
|----|----|
| AB | BC |
| BA | CB |
| AC | BD |
| CA | DB |
| AD | CD |
| DA | DC |



$k=3$

4 choices \times 3 choices \times 2 choices
 $4 \cdot 3 \cdot 2 = 24$ possibilities

Examples of k -permutations

(Problem 7, page 84) How many 9-digit numbers can be made from the digits 1, 2, 3, 4, 5, 6, 7, 8, 9 if repetition is not allowed and all the odd digits occur first, followed by all the even digits.

1, 3, 5, 7, 9 2, 4, 6, 8
5 odd digits 4 even digits
 $5! = 120$ 5 digit permutations $4! = 24$ 4 digit permutations here.

5 digits	4 digits
120 choices	24 choices

$$\text{Total} = 120 \cdot 24 = \begin{array}{r} 2400 \\ + 480 \\ \hline 2880 \end{array}$$

Remark: If we just wanted sequences of digits w/o repetition, we'd have $9!$ possibilities.

(Problem 15, page 84) In a club of 15 people, there is a president, vice-president, secretary, and treasurer. In how many different ways can this be done?

$$\begin{aligned}
 & \text{PRES} \quad \text{VP} \quad \text{SEC} \quad \text{TREAS} \\
 & 15 \cdot 14 \cdot 13 \cdot 12 = 15 \cdot 14 \cdot 13 \cdot 12 \\
 & = P(15, 4) \\
 & = \frac{15!}{11!} = \frac{15 \cdot 14 \cdot 13 \cdot 12 \cdot \cancel{11!}}{11! \cdot \cancel{11!}}
 \end{aligned}$$

1, 2, 3, 4, - - - - - , 11, 12, 13, 14, 15

