Permutations

$4! = 1\cdot 2\cdot 3\cdot 4 = 24$

Factorials

$\boxed{0! = 1}$

$1! = 1 \quad 2! = 2\cdot 1 = 2$

$3! = 1\cdot 2\cdot 3 = 6$

Definition 1: For $n \in \mathbb{Z}$, $n \geq 0$, define $0! = 1$ and $n! = (1)(2)\cdots(n-1)(n)$. Alternatively, define $n!$ for non-negative integers n by setting $0! = 1$ and $n! = n(n-1)!$.

Proposition: The number of different lists of length n made up of elements from the set $\{1, 2, \ldots, n\}$, *without repetitions*, is $n!$.

$\{1,2\}$ — $(1,2)$, $(2,1)$ — 2 lists

$X = \emptyset \quad () \quad 1$

$\{1,2,3\}$ — $(1,2,3)$, $(1,3,2)$, $(2,1,3)$, $(2,3,1)$, $(3,1,2)$, $(3,2,1)$ — 6 lists

Count:
n choices for first position
$n-1$ second
$n-2$ third
$\vdots$
1 choice

$= n!$ is product

Proof: Case $n=1$. $X = \{1\}$ There is only 1 list (1),

must show:

If # of lists of length $n-1$ is $(n-1)!$ $\Longrightarrow$ # of lists of length n is $n!$

$X = \{1, \ldots\ldots, n\}$.

List of length n $\longleftrightarrow$ list of length $n-1$ + final element.

Lists of length n divide up into n groups; $S_1, \ldots, S_n$

$S_i = \{$lists with i in last position$\}$.

[... | 1]
[... | 2]
$\vdots$
[... | n]

# of lists $= |S_1| + |S_2| + \cdots + |S_n|$

$|S_1|$ consists of a list of length $n-1$ using numbers $\underbrace{2, \ldots, n}_{n-1 \text{ numbers}}$.

so $|S_1|$ has $(n-1)!$ elements

Similarly each $|S_i|$ has $(n-1)!$ elements

Total # $= |S_1| + |S_2| + \cdots + |S_n| = n\cdot(n-1)! = n!$

That's what we wanted to show.

$X = \{1,2,3\}$

2	3	1
3	2	1
1	3	2
3	1	2
1	2	3
2	1	3

Permutations

Definition: Let X be a set. A permutation of X is a list of length $|X|$ of the elements of X, without repetition. (**Note:** There are other definitions of permutations in other contexts, all related to this one).

Examples:

Permutation of $\{2,3,4\}$ is a list $(2,3,4)$

there are $6 = 3!$ such permutations

3, 2, 4

2, 4, 3

4, 2, 3

4, 3, 2

3, 4, 2

$X = \{A, B, C, D\}$

permutations of X:

$$\left.\begin{array}{l} ABCD \\ BACD \\ CABD \\ \vdots \end{array}\right\} \begin{array}{l} 24 \text{ permutations} \\ 24 = 4! \end{array}$$

Definition: Let X be a set. A k-permutation of X is a list of k elements of X without repetition. $P(n,k)$ is the number of k permutations of a set with n elements.

Proposition: The number $P(n,k)$ of k-permutations of a set with n elements is $\underbrace{n(n-1)\cdots(n-k+1)}_{\text{k terms in this product}}$ or, equivalently

$$\underbrace{P(n,k) =}\frac{n!}{(n-k)!}$$

Example: $X = \{A, B, C, D\}$
$n = 4$
$K = 2$

AB BC
BA CB
AC BD
CA DB
AD CD
DA DC

$\boxed{12}$

$K = 3$
4 choices $\times$ 3 choices $\cdot$ 2 choices
$4 \cdot 3 \cdot 2 = 24$ possibilities

Proof:

$$\frac{n!}{(n-k)!} = \frac{1 \cdot 2 \cdot \cdots \cdot n}{1 \cdot 2 \cdots (n-k)} = (n-k+1) \cdots n$$

Multiplication principle.

n choices for 1^{st} slot
$n-1$ choices for 2^{nd} slot
$\vdots$
$n-k+1$ choices for k^{th} slot

$n(n-1) \cdots (n-k+1)$

Examples of k-permutations

(Problem 7, page 84) How many 9-digit numbers can be made from the digits $1, 2, 3, 4, 5, 6, 7, 8, 9$ if repetition is not allowed and all the odd digits occur first, followed by all the even digits.

1, 3, 5, 7, 9

2, 4, 6, 8

5 odd digits

4 even digits

$5! = 120$ 5 digit permutations

$4! = 24$ 4 digit permutations here.

5 digits	4 digits
120 choices	24 choices

$$\text{Total} = 120 \cdot 24 = \begin{array}{r} 2400 \\ +\ 480 \\ \hline 2880 \end{array}$$

<u>Remark</u>: If we just wanted sequences of digits w/o repetition, we'd have $9!$ possibilities.

(Problem 15, page 84) In a club of 15 people, there is a president, vice-president, secretary, and treasurer. In how many different ways can this be done?

PRES VP SEC TREAS

$$15 \cdot 14 \cdot 13 \cdot 12 = 15 \cdot 14 \cdot 13 \cdot 12$$

$$= P(\underline{15}, \underline{4})$$

$$= \frac{15!}{11!} = \frac{15 \cdot 14 \cdot 13 \cdot 12 \cdot \cancel{11} \cdot \cancel{\cdot} \cancel{\cdot}}{\cancel{11} \cdot \cancel{10} \cdot \cancel{\cdot} \cdots}$$

1, 2, 3, 4, - - - - - -, 11, 12, 13, 14, 15