Counting

Lists

Definition: a (finite) list is an element of the Cartesian product of sets $X = X_1 \times \cdots X_n$. A common counting problem is to determine the number of lists with certain properties whose entries are drawn from a Cartesian product like X.

(me, you, fred, jan, ..., millie)

Multiplication Principle

> **Fact 3.1 (Multiplication Principle)** Suppose in making a list of length n there are a_1 possible choices for the first entry, a_2 possible choices for the second entry, a_3 possible choices for the third entry, and so on. Then the total number of different lists that can be made this way is the product $a_1 \cdot a_2 \cdot a_3 \cdots \cdot a_n$.

Figure 1: Multipication Principle (p. 69)

This informal principle can be applied in many settings, although in most cases there is a *hidden* proof by induction.

Example

Proposition: Suppose that $X_1, \ldots, X_n$ are finite sets. Then

$$|X_1 \times \cdots \times X_n| = |X_1||X_2| \cdots |X_n|.$$

$X_1 = \{a, b\}$ $X_2 = \{1, 2\}$ $X_1 \times X_2 = \{(a,1), (a,2), (b,1), (b,2)\}$

$|X_1 \times X_2| = 4$

Proof: First suppose $n = 2$. We need to show that $|X_1 \times X_2| = |X_1||X_2|$

$n = |X_1|$ $m = |X_2|$

X_1: $x_{11}, \ldots, x_{1n}$; x_{21} x_{22} ... x_{2m}; (x_{11}, x_{21})

$n \cdot m$ total possibilities

Look at $X_1 \times X_2 \times \cdots X_{n+1} = \{(a_1, a_2, \ldots, a_{n+1}) \mid a_i \in X_i \quad i = 1, \ldots, n+1\}$

Look at this as $(X_1 \times \cdots \times X_n) \times X_{n+1}$

Look at $(a_1, \ldots, a_n, a_{n+1})$ as $((a_1, \ldots, a_n), a_{n+1})$

$|X_1 \times \cdots X_{n+1}| = |(X_1 \times \cdots X_n) \times X_{n+1}|$

$|X_1 \times \cdots \times X_n| = a_1 \cdots a_n$ and the 2-case says

$|(X_1 \times \cdots \times X_n) \times X_{n+1}| = a_1 \cdots a_n \cdot a_{n+1}$

Example

How many ways can you order a coffee if your choices are whole, skim, or soy milk; small, medium, or large size; and one or two shots of espresso?

$X_1 = \{\text{whole, skim, soy}\}$

$X_2 = \{\text{small, medium, large}\}$

$X_3 = \{\text{one shot, 2 shots}\}$

type of coffee $\longleftrightarrow$ an elt of $X_1 \times X_2 \times X_3$

$|X_1 \times X_2 \times X_3| = 3 \cdot 3 \cdot 2 = \underline{18}$

Example

Consider lists of length 4 made with the symbols A, B, C, D, E, F, G.

Question: How many lists are there made up of these symbols (no special conditions)

Count elements in the Cartesian Product

$Y \times Y \times Y \times Y$ where $Y = \{A, B, C, D, E, F, G\}$,

$|Y| = 7$

$|Y \times Y \times Y \times Y| = 7^4$

Example continued lists of length 4 $Y = \{A, B, C, D, E, F, G\}$.

Question: How many lists are there if no letter is repeated?

Lists of length 1: 7 possibilities.

2: Break up list by 1^{st} letter:

7 possible 1^{st} letters

A _ 6 possible lists
B _ 6 possible here
C _ ⋮
⋮
G _ 6 possible here

$7 \cdot 6 = 42$

Lists of length 3

42 lists of length 2

AB _ 5 lists begin AB
AC _ ⋮
AD _
⋮
GA _ 5 lists begin GA

$42 \cdot 5$ lists of length 3 with no repetitions

length 4:

$42 \cdot 5$ lists of length 3

ABC _ 4 possible 4th letters
ABD _
⋮

$42 \cdot 5$ lists of length 3

4 4th letters

$42 \cdot 5 \cdot 4$

$7 \cdot 6 \cdot 5 \cdot 4$.

Example continued

length 4, $Y = \{A, B, C, D, E, F, G\}$

Question: How many lists are there length 4, if there are no repetitions, and at least one of the letters is an E?

Split Lists up according to where the "E" is.

group 1: E in position 1

E_ _ _. Next 3 are a list of length 3, no repetitions, letters are A, B, C, D, F, G 6 letters

$6 \cdot 5 \cdot 4$

group 2 E in position 2

_ E _ _ $6 \cdot 5 \cdot 4$ possible

group3 E in position 3

_ _ E _ $6 \cdot 5 \cdot 4$

group 4 E in position 4

_ _ _ E $6 \cdot 5 \cdot 4$

Total

$(6 \cdot 5 \cdot 4) 4$ lists.

Example continued

Question: How many lists are there if repetition *is* allowed, and the list contains at least one E?

All of these lists belong to $Y \times Y \times Y \times Y$

$Y \times Y \times Y \times Y$

7^4

elements

lists of length 4
But NO E's

6^4

lists of length 4
at least one E

$7^4 - 6^4$

$7^4 \quad = \quad$ # elements here $+$ # elts here

$|X \times X \times X \times X| = 6^4$

$X \times X \times X \times X$

$X = \{A, B, C, D, F, G\}$