

Counting

Lists

Definition: a (finite) list is an element of the Cartesian product of sets $X = X_1 \times \dots \times X_n$. A common counting problem is to determine the number of lists with certain properties whose entries are drawn from a Cartesian product like X . (me, you, fred, jan, ..., millie)

Multiplication Principle

Fact 3.1 (Multiplication Principle) Suppose in making a list of length n there are a_1 possible choices for the first entry, a_2 possible choices for the second entry, a_3 possible choices for the third entry, and so on. Then the total number of different lists that can be made this way is the product $a_1 \cdot a_2 \cdot a_3 \cdot \dots \cdot a_n$.

Figure 1: Multiplication Principle (p. 69)

This informal principle can be applied in many settings, although in most cases there is a *hidden* proof by induction.

Example

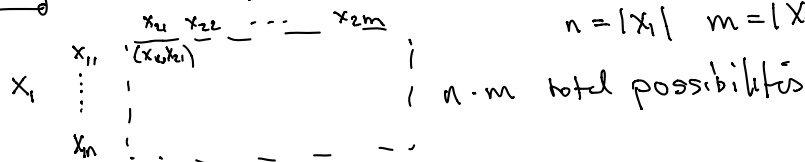
Proposition: Suppose that X_1, \dots, X_n are finite sets. Then

$$|X_1 \times \dots \times X_n| = |X_1| |X_2| \dots |X_n|.$$

$$X_1 = \{a, b\} \quad X_2 = \{1, 2\} \quad X_1 \times X_2 = \{(a, 1), (a, 2), (b, 1), (b, 2)\}$$

$$|X_1 \times X_2| = 4$$

Proof: First suppose $n=2$. We need to show that $|X_1 \times X_2| = |X_1| |X_2|$



$$\text{look at } X_1 \times X_2 \times \dots \times X_{n+1} = \{(a_1, a_2, \dots, a_{n+1}) \mid a_i \in X_i \ i=1, \dots, n+1\}$$

$$\text{look at this as } (X_1 \times \dots \times X_n) \times X_{n+1}$$

$$\text{Look at } (a_1, \dots, a_n, a_{n+1}) \text{ as } ((a_1, \dots, a_n), a_{n+1})$$

$$|X_1 \times \dots \times X_{n+1}| = |(X_1 \times \dots \times X_n) \times X_{n+1}|$$

$$|X_1 \times \dots \times X_n| = a_1 \cdot \dots \cdot a_n \quad \text{and the 2-case says } |(X_1 \times \dots \times X_n) \times X_{n+1}| = a_1 \cdot \dots \cdot a_n \cdot a_{n+1}$$

Example

How many ways can you order a coffee if your choices are whole, skim, or soy milk; small, medium, or large size; and one or two shots of espresso?

$$X_1 = \{\text{whole, skim, soy}\}$$

$$X_2 = \{\text{small, medium, large}\}$$

$$X_3 = \{\text{one shot, 2 shots}\}$$

type of coffee \leftrightarrow an elt of $X_1 \times X_2 \times X_3$

$$|X_1 \times X_2 \times X_3| = 3 \cdot 3 \cdot 2 = \underline{18}$$

Example

Consider lists of length 4 made with the symbols A, B, C, D, E, F, G .

Question: How many lists are there made up of these symbols (no special conditions)

Count elements in the Cartesian Product

$$Y \times Y \times Y \times Y \quad \text{where } Y = \{A, B, C, D, E, F, G\},$$

$$|Y| = 7$$

$$|Y \times Y \times Y \times Y| = 7^4$$

Example continued

lists of length 4
 $Y = \{A, B, C, D, E, F, G\}$.

Question: How many lists are there if no letter is repeated?

Lists of length 1: 7 possibilities.

2: Break up list by 1st letter:

7 possible 1st letters {
A - 6 possible lists
B - 6 possible here
C -
⋮
G - 6 possible here } 7 · 6 = 42

List of length 3

42 lists of length 2 {
AB - 5 lists begin AB
AC -
AD -
⋮
GA - 5 lists begin GA } 42 · 5
lists of length 3 with no repetitions

length 4: 42 · 5 lists of length 3 {
ABC - 4 possible 4th letters
ABD -
⋮ } 42 · 5 · 4
4th letters
7 · 6 · 5 · 4.

Example continued

length 4, $\gamma = \{A, B, C, D, E, F, G\}$

Question: How many lists are there if there are no repetitions, and at least one of the letters is an E?

Split Lists up according to where the "E" is.

group 1: E in position 1

E _ _ _

Next 3 are a list of length 3, no repetitions letters are A, B, C, D, F, G 6 letters
6 · 5 · 4

group 2 E in position 2

_ E _ _

6 · 5 · 4 possible

group 3 E in position 3

_ _ E _

6 · 5 · 4

group 4 E in position 4

_ _ _ E

6 · 5 · 4

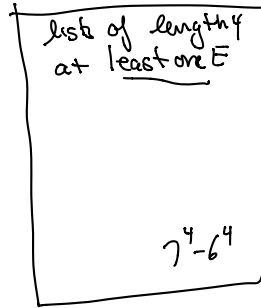
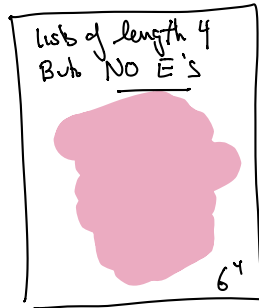
Total
(6 · 5 · 4) · 4
lists.

Example continued

Question: How many lists are there if repetition *is* allowed, and the list contains at least one *E*?

All of these lists belong to $Y \times Y \times Y \times Y$

$Y \times Y \times Y \times Y$
 7^4
 elements



$7^4 =$

elements + # elts
 here $1 \times \dots \times 1 = 6^4$ here
 $X \times X \times X \times X$
 $X = \{A, B, C, D, F, G\}$