Quantifiers

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Remember that an open sentence is a sentence that includes variables; when you specify the variables, the open sentence becomes a statement that can be true or false.

Open sentence:

$x > 5$

$x = 7$ TRUE

$x = 3$ FALSE

$xy = 1$

$x = \frac{1}{2}, y = 2$

TRUE

$x = \frac{1}{2}, y = 5$

FALSE

Quantifiers

← find values of x making the open sentence a true statement

Most equations that we want to "solve" are really open sentences. For example,

$$\begin{aligned} 3x &= 7 \\ x^2 + 5x + 6 &= 0 \end{aligned}$$

are open sentences whose truth depends on the choice of x.

Whether or not these equations even *have* solutions depends on what kind of values x is allowed to have.

Quantifiers

$$\left.\begin{aligned} &3x = 7 \\ &\cancel{8x}\ x^2 + 5x + 6 = 0 \end{aligned}\right\} (-2, -3)$$

For example:

- neither of these equations have solutions if x is required to be a natural number.
- if x is allowed to be an integer, then the second equation has two solutions, but the first one still has none.

  $3x = 7$ has no integer solutions

- if x is allowed to be a rational number, then both equations have solutions.

Quantifiers

Quantifiers are an element of the logical language that put a scope on the possible values of a variable in an open sentence, and in the process convert the open sentence into a statement.

The are two quantifiers: - "there exists" makes the statement about *some* x in a particular set, - "for all" makes the statement about *all* x in a particular set.

open sentence true for some $x \in A$ there exists
open sentence true for all $x \in A$ for all

where is a given set

Existential quantifier (there exists)

for at least one.

"There exists $x \in \mathbb{Q}$ such that $3x = 7$"

This statement is true if and only if the subset

$$X = \{x : x \in \mathbb{Q}, 3x = 7\}$$

has at least one element – there is *some* x so that $3x = 7$ among the $x \in \mathbb{Q}$.

$\exists x, P(x)$

- "There exists $x \in \mathbb{Q}$ such that $3x = 7$" is True
- "There exists $x \in \mathbb{Z}$ such that $3x = 7$" is False

More generally, if X is any set, and $P(x)$ is an open sentence, then the statement "There exists $x \in X$ so that $P(x)$" (in symbols "$\exists x, P(x)$") is true exactly when the set

$\exists x \in X, P(X)$

$$Y = \{x : x \in X, P(x)\}$$

has at least one element.

Univeral quantifier (for all)

"For all $x \in \mathbb{N}$ $x^2 > 0$"

The statement "For all $x \in \mathbb{N}$, $x^2 > 0$" is true if and only if

$$X = \{x : x \in \mathbb{N}, x^2 > 0\} = \mathbb{N}.$$

every natural number x has $x^2 > 0$

It claims something is true for *all* $x \in \mathbb{N}$. This is in fact a true statement.

On the other hand, the statement "For all $x \in \mathbb{Z}$, $x\cancel{\wedge}2>0$" is false since $0^2 = 0$ and $0 \in \mathbb{Z}$.

x^2

More generally, the statement "For all $x \in X$, $P(x)$" (in symbols "$\forall x, P(x)$") is true exactly when

$$X = \{x \in X : P(x)\}.$$

$(\forall x \in \mathbb{Z}, x^2 > 0)$ is a false statement

This is a statement about *all* $x \in X$.

A few more examples

$x^2 = 15$

- There exists $x \in \mathbb{R}$ such that $x^2 = 15$. TRUE: $x = \sqrt{15}$
- For all $y \in \mathbb{R}$, $|\sin(y)| \leq 1$. [For all $y \in \mathbb{R}$, $|\sin(y)| \leq 1$] TRUE
- There exists a subset X of $\mathbb{N}$ which has 5 elements.

$\exists X \in \wp(\mathbb{N})$ such that $|X| = 5$ true

$X = \{1, 2, 3, 4, 5\}$ is an example

Negating quantified statements

$\sim(\exists x, P(x)) = (\forall x, \sim P(x))$

The statement "There exists $x \in X$ such that $P(x)$" is false exactly when "For all $x \in X$, not $P(x)$" is true.

No x in P(x) with P(x) true
All x in P(x) have P(x) FALSE
All x in P(x) have ~P(x) TRUE

For example, "There exists $x \in \mathbb{R}$ such that $x^2 < 0$" is false because "For all $x \in \mathbb{R}$, $x^2 \geq 0$" is true.

The statement "For all x, $P(x)$" is false exactly when "There exists x such that not $P(x)$" is true.

All x have P(x) is FALSE
SOME x has P(x) False
SOME x has NOT P(x) TRUE

For example, the statement "For all $x \in \mathbb{N}$, $x^2 > 0$" is true because "There exists $x \in \mathbb{N}$ with $x^2 \leq 0$." is false.

$\sim(\forall x, P(x)) = (\exists x, \sim P(x))$

Existence and "OR", For all and "AND"

There exists $x \in X$ such that $P(x)$ is a kind of "OR" statement.

For all $x \in X$ such that $P(x)$ is a kind of "AND" statement.

X open sentence $P(x)$

$x \in \mathbb{N}$.

$2x = 6$

there exists $x \in \mathbb{N}$ so that $2x = 6$.

$[2 \cdot 1 = 6$ OR $2 \cdot 2 = 6$ OR $2 \cdot 3 = 6$ OR $\cdots]$

For all $\Longleftrightarrow$ "AND"

For all $x \in \mathbb{N}$, $x^2 > 0$.

$1^2 > 0$ AND $2^2 > 0$ AND $3^2 > 0$ AND $\cdots$