Problem 12.2.10

Problem: Prove that the function $f : \mathbb{R} - \{1\} \to \mathbb{R} - \{1\}$ defined by

$$f(x) = \left(\frac{x+1}{x-1}\right)^3$$

is bijective.

We must show that f is both injective and surjective. So let's consider injectivity first. Assume $a, b \in \mathbb{R} - \{1\}$ and that $f(a) = f(b)$.

$$\left(\frac{a+1}{a-1}\right)^3 = \left(\frac{b+1}{b-1}\right)^3$$

We will show that this implies that $a = b$.

Step 1. $$\left(\frac{a+1}{a-1}\right)^3 = \left(\frac{b+1}{b-1}\right)^3$$

Suppose $u^3 = v^3$. Then $u = v$. This is the assertion that $f(x) = x^3$ is injective. How do we know that $f(x) = x^3$ is injective.

So taking cube roots,

$$\left(\frac{a+1}{a-1}\right) = \left(\frac{b+1}{b-1}\right)$$

$$(a+1)(b-1) = (b+1)(a-1)$$

$$\cancel{ab} - a + b \cancel{-1} = \cancel{ab} + a - b \cancel{-1}$$

$$2b = 2a$$

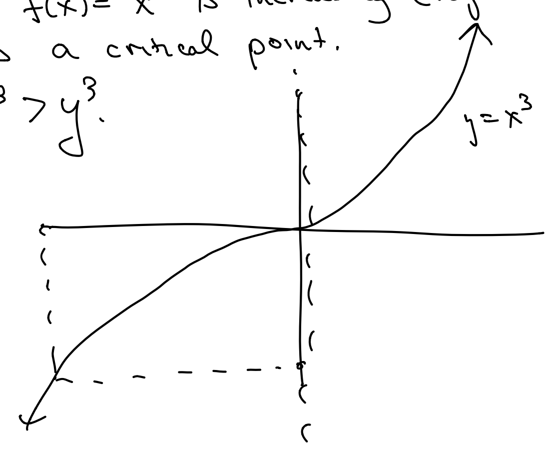
$$b = a.$$

Therefore f is injective.

Lemma: $f(x) = x^3$ is injective

Proof: Observe that $f'(x) = 3x^2 \geq 0$ for all x and it's bigger than zero unless $x = 0$. $f(x) = x^3$ is increasing everywhere except at zero where it has a critical point.

So if $x > y$ then $x^3 > y^3$.



Suppose x, y are not equal. We can assume $x > y$. Then $x^3 > y^3$ so $x^3 \neq y^3$.

$f(x) = x^3$ is injective.

Surjectivity. Given $b \in \mathbb{R} - \{1\}$ we need to find an $a \in \mathbb{R} - \{1\}$ so that

$b \neq 1$.

$$\left(\frac{a+1}{a-1}\right)^3 = b$$

$$\left(\frac{a+1}{a-1}\right) = \sqrt[3]{b}$$

$f(x) = x^3$ is surjective

$x^3 = b$ has a solution for all $b \in \mathbb{R}$, namely $\sqrt[3]{b}$.

$$a+1 = \sqrt[3]{b}\,(a-1)$$

$$a - (\sqrt[3]{b})a = -\sqrt[3]{b} - 1$$

$$a(1 - \sqrt[3]{b}) = -\sqrt[3]{b} - 1$$

$$a = \frac{-\sqrt[3]{b} - 1}{1 - \sqrt[3]{b}} = \frac{\sqrt[3]{b} + 1}{\sqrt[3]{b} - 1}$$

Prop: The composition of bijective functions is bijective.

$f(x) = x^3$ bijective $\quad f(g(x))$ is bijective.

$g(x) = \dfrac{x+1}{x-1}$ bijective