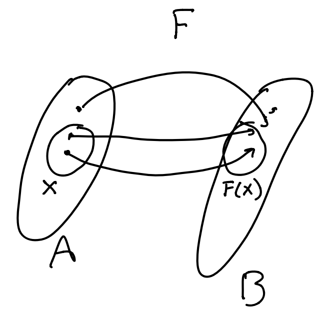
Image and preimage

Image and preimage

Key defintions

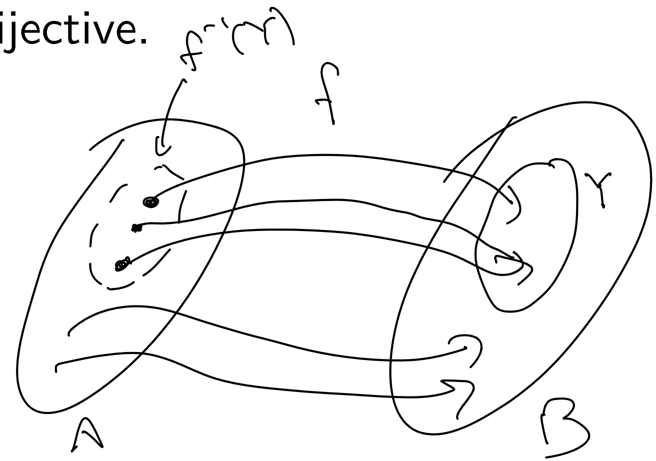


Definition: Let $f : A \to B$ be a function.

1. If $\boxed{X \subseteq A}$, then the **image** of X is the set $f(X) = \{f(x) : x \in X\} \subseteq B$.
2. If $Y \subseteq B$, then the **preimage** of Y is the set $f^{-1}(Y) = \{x \in A : f(x) \in Y\} \subseteq A$.

$f(A)$ = range

Note: $f^{-1}(Y)$ is defined *even when* f^{-1} is not a function, i.e. even when f is not bijective.



Example 12.13

Example 12.13 Let $f : \{s,t,u,v,w,x,y,z\} \to \{0,1,2,3,4,5,6,7,8,9\}$ be

$$f = \{(s,4),(t,8),(u,8),(v,1),(w,2),(x,4),(y,6),(z,4)\}.$$

This f is neither injective nor surjective, so it certainly is not invertible. Be sure you understand the following statements.

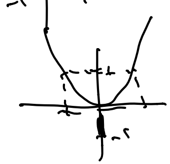
1. $\boxed{f(\{s,t,u,z\}) = \{8,4\}}$
2. $f(\{s,x,z\}) = \{4\}$ $\quad \{4\} \ (\neq 4)$
3. $\boxed{f(\{s,v,w,y\}) = \{1,2,4,6\}}$
4. $\boxed{f(\emptyset) = \emptyset}$
5. $f^{-1}(\{4\}) = \{s,x,z\}$ $\quad f^{-1}(\{4\}) = \{s,x,z\}$
6. $f^{-1}(\{4,9\}) = \{s,x,z\}$ $\quad f^{-1}(\{4,9\}) = \{s,x,z\}$
7. $f^{-1}(\{9\}) = \emptyset$
8. $f^{-1}(\{1,4,8\}) = \{s,t,u,v,x,z\}$

$$f(\{s,t,u,v,w,x,y,z\}) = \{4,8,1,2,6\} = \text{range}(f)$$

$f : \mathbb{R} \to \mathbb{R} \qquad f(x) = x^2$

$X = [-1,1]. \qquad f(X) = \{x^2 : x \in [-1,1]\} = [0,1]$

$f^{-1}([-1,1]) = \{x \in \mathbb{R} \mid x^2 \in [-1,1]\} = [-1,1]$



$f^{-1}([0,1]) = [-1,1] \quad \le x^2 \le 1 \Rightarrow 0 \le x^2 \le 1 \quad -1 \le x \le 1$

Problem 12.6.7

Problem: Prove that, if $f : A \to B$ is a function, and W and X are subsets of A, then

$$\underline{f(W \cap X)} \subseteq \underline{f(W) \cap f(X)}$$

$$\underline{f(W \cap X)} = \{ f(a) \mid a \in W \cap X \}.$$

$$= \{ f(a) \mid a \in \cancel{\ } W \text{ and } a \in X \}.$$

$f(W) = \{ f(a) \mid a \in W \}$ $\qquad$ $f(X) = \{ f(a) \mid a \in X \}.$

If $y = f(a)$ for some $a \in W \cap X$ then $y = f(a)$ for some $a \in W$
($y \in f(W \cap X)$) $\qquad$ and $y = f(a)$ for some $a \in X$.

This is true. $a \in W \cap X$ means $a \in W$ and $a \in X$.

Problem 12.6.9

Problem: Prove that, if $f : A \to B$ is a function, and W and X are subsets of A then

$$f(W \cup X) = f(W) \cup f(X)$$

$f(W \cup X) = \{f(a) \mid a \in W \cup X\}$
$f(W) = \{f(a) \mid a \in W\}$
$f(X) = \{f(a) \mid a \in X\}$.

MUST SHOW:
1. $f(W \cup X) \subseteq f(W) \cup f(X)$

and $f(W) \cup f(X) \subseteq f(W \cup X)$.

1. Assume $x \in f(W \cup X)$. Then $x = f(a)$ for some $a \in W \cup X$ or in other words some $a \in W$ or $a \in X$.

Therefore $x = f(a)$ for some $a \in W$ OR $x = f(a)$ for some $a \in X$.

$x \in f(W)$ or $x \in f(X)$.

$x \in f(W \cup X) \Rightarrow x \in f(W)$ or $x \in f(X) \Leftrightarrow$
$x \in f(W) \cup f(X)$.

2. $x \in f(W)$ or $x \in f(X)$
$x \in f(W)$ means $x = f(a)$ for some $a \in W$,
$x \in f(X)$ means $x = f(a)$ for some $a \in X$.

$x = f(a)$ for some $a \in W$ or $a \in X$.

$x = f(a)$ for some $a \in W \cup X$

$x \in f(W \cup X)$.