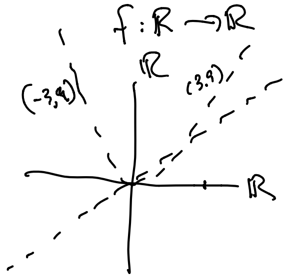
Inverse functions

Inverse functions

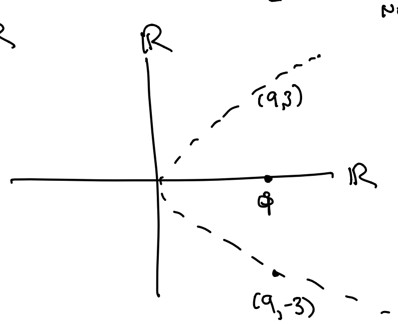
Let A and B be sets and let $f \subset A \times B$ be a function ($f : A \to B$ in the alternative notation). Since f is a relation, one can consider the inverse relation f^{-1}.

$f : \mathbb{R} \to \mathbb{R}$ $\quad f(x) = x^2$ $\quad f = \{(x, x^2) \mid x \in \mathbb{R}\}$



$f^{-1} = \{(x^2, x) \mid x \in \mathbb{R}\}$

Note: $y = \sqrt{x}$, but really $x = y^2$



$(9,3) \in f^{-1}$
$(9,-3)$

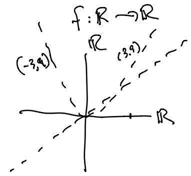
same "x-coordinate"
but different y-coordinates.

Sometimes the inverse relation f^{-1} *is* a function, and sometimes it *is not* a function.

Examples

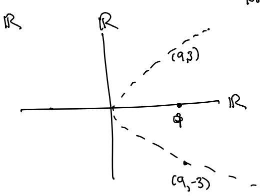
Let R be the relation $\{(x, x^2) : x \in \mathbb{R}\} \subset \mathbb{R} \times \mathbb{R}$.

$f : \mathbb{R} \to \mathbb{R}$ $\quad f(x) = x^2 \quad f = \{ (x, x^2) \mid x \in \mathbb{R} \}$



$f^{-1} = \{ (x^2, x) \mid x \in \mathbb{R} \}$

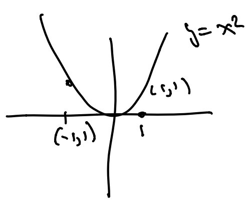
Note: $y = \sqrt{x}$, but really $x = y^2$



$(9,3) \in f^{-1}$
$(9,-3)$
same "x-coordinate" but different y-coordinates.

- R is a function because for every $x \in \mathbb{R}$ there is a unique $y = x^2$ in $\mathbb{R}$ so that $(x, y) \in R$.

- R^{-1} is *not* a function because both $\underline{(1, -1)}$ and $\underline{(1, 1)}$ are in R^{-1}.

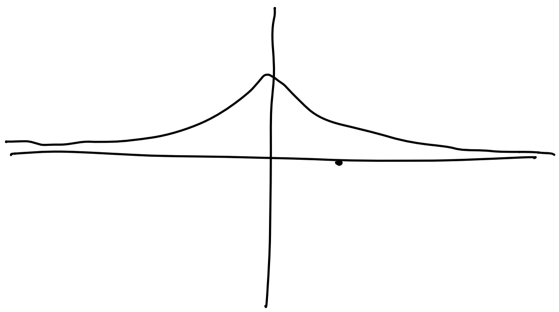


$(1,1)$, $(1,-1)$ are in R^{-1}
NOT a function.

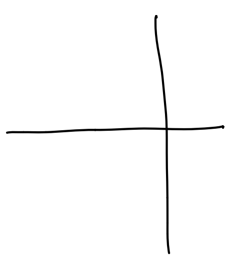
Example

Let R be the relation $\{(x, \frac{1}{1+x^2}) : x \in \mathbb{R}\} \subset \mathbb{R} \times \mathbb{R}$.

- R is a function.



- R^{-1} is *not* a function because $0 < \frac{1}{1+x^2} \leq 1$ for all x, and therefore there is no pair $(x, y) \in R^{-1}$ with $x = 2$.



$f = \{(x,y) \mid 0 \leq y \leq 1\}$

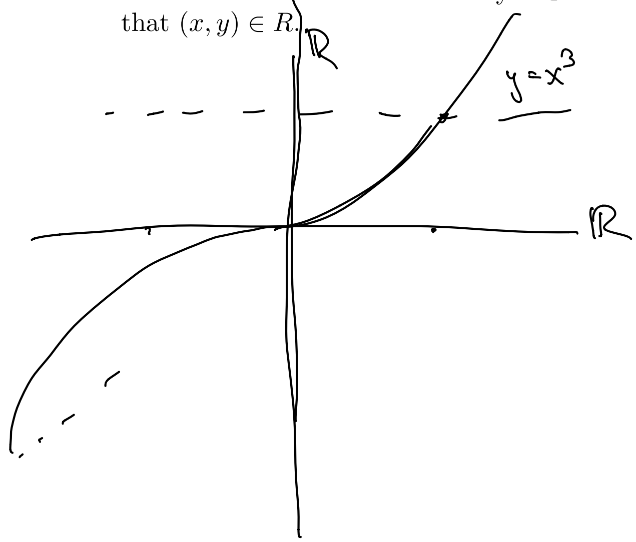
$f^{-1} = \{(y,x) \mid 0 < y \leq 1\}$

Can't be a function with domain $\mathbb{R}$ because the first coordinates don't cover $\mathbb{R}$.

Examples

Let R be the relation $\{(x, x^3) : x \in \mathbb{R}\} \subset \mathbb{R} \times \mathbb{R}$.

- R is a function because for every $x \in \mathbb{R}$ there is a unique $y = x^3$ in $\mathbb{R}$ so that $(x, y) \in R$.



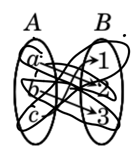
- R^{-1} is also a function because for every $x \in \mathbb{R}$ there is a unique $y = x^{1/3}$ for every $x \in \mathbb{R}$ so that $(x, y) \in R^{-1}$.

$f = \{(x, y) \mid y = x^3\}$

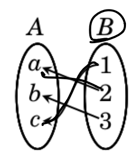
$f^{-1} = \{\overset{y,x}{\cancel{(x,y)}} \mid y = x^3 \text{ or } x = y^{1/3}\}$

For every real number there is exactly one cube root.

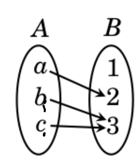
Examples (p. 239)



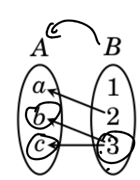
$f = \{(a,2),(b,3),(c,1)\}$



$f^{-1} = \{(2,a),(3,b),(1,c)\}$



$g = \{(a,2),(b,3),(c,3)\}$



$g^{-1} = \{(2,a),(3,b),(3,c)\}$

The Inverse Function Theorem

$F: A \to B$ $\quad F^{-1}:$

Theorem: Let $F \subset A \times B$ be a function. The inverse relation $F^{-1} \subset B \times A$ is also a function if and only if F is bijective.

F^{-1} a function $\Rightarrow$ F injective and F surjective

F inj and surj $\Rightarrow F^{-1}$ a function

Proof: Suppose F^{-1} is a function.

That means $F^{-1} = \{(b,a) \mid (a,b) \in F\}$

has the property that, for all $b \in B$, there is a unique $a \in A$ so that $(b,a) \in F^{-1}$.

We first show F is injective. Suppose $a, a' \in A$ and $F(a) = F(a')$. Let $b = F(a)$ and $b' = F(a')$. Then (a,b) and (a',b') are in F. So (b,a) and (b',a') are in F^{-1} and $b = b'$. F^{-1} is a function, $(b,a) \in F^{-1}$ and $(b,a') \in F^{-1}$ then $a = a'$. Therefore $(F(a) = F(a')) \Rightarrow a = a'$ so F is injective.

Now we show F is surjective. Let $b \in B$. We must find $a \in A$ so that $F(a) = b$ or $(a,b) \in F$. Now F^{-1} a function so there is an ordered pair $(b,x) \in F^{-1}$, so $(x,b) \in F$ so $F(x) = b$ so $a = x$ is our desired element of A.

Suppose F is bijective. We must show: given $b \in B$, there is at least one pair (b,a) for some $a \in A$ that belongs to F^{-1}. $(b,a) \in F^{-1}$ means $(a,b) \in F$. Since F is surjective, there is an $a \in A$ so that $(a,b) \in F$, so $(b,a) \in F^{-1}$ as desired.

Now suppose (b,a) and (b,a') are F^{-1}. Then (a,b) and (a',b) are in F. But F is injective and this says $F(a) = b = F(a')$. So $a = a'$ and $(b,a) = (b,a')$. So there is only one pair with b in its first coordinate.

Inverse functions (definition)

Definition: If $f : A \to B$ is bijective, then its **inverse** is the function

$$\underline{f^{-1} : B \to A.}$$

We have

$$f^{-1} \circ f : A \to A = i_A.$$

and

$$f \circ f^{-1} : B \to B = i_B$$

$$f : A \xrightarrow{f} B \xrightarrow{f^{-1}} A \qquad (i_A)$$

$$B \xrightarrow{f^{-1}} A \xrightarrow{f} B \qquad (i_B)$$

Compute

$(f^{-1} \circ f)(a)$:

$(a, \underline{f(a)}) \in f$

$(\underline{f(a)}, a) \in f^{-1}$

show that $f^{-1} \circ f = i_A$

$(b, \underline{f^{-1}(b)}) \in f^{-1}$

$(\underline{f^{-1}(b)}, b) \in (f^{-1})^{-1} = f$

$(f^{-1})^{-1} = f$

$f^{-1} = \{(b,a) \mid (a,b) \in f\}$

$(f^{-1})^{-1} = \{(a,b) \mid (a,b) \in f\}$

$f \circ f^{-1} = i_B.$

If $B = A$

$f \circ f^{-1} = f^{-1} \circ f = i_A.$