Inverse Relations and Inverse Functions

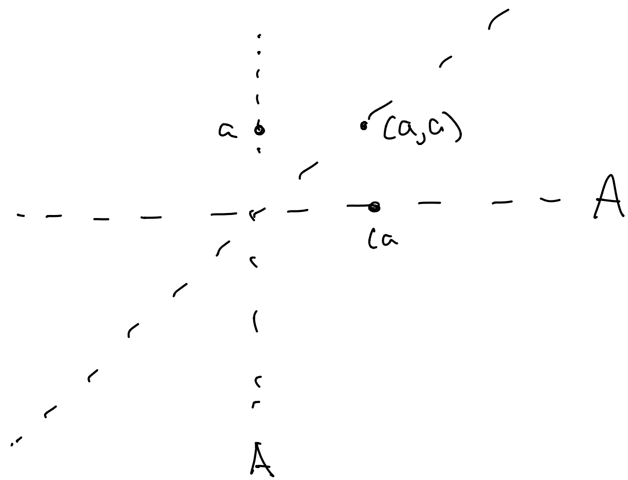
The identity function

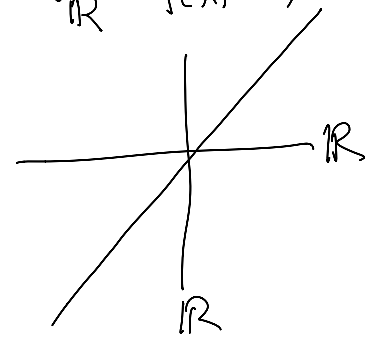
Definition: Let A be a set. The identity function $i_A : A \to A$ is the function defined by $i_A(x) = x$ for all $x \in A$.

As a set of ordered pairs, $i_A \subset A \times A$ consists of all pairs (a, a) for $a \in A$.

$$i_A = \{(a,a) \mid a \in A\}$$

$i_{\mathbb{R}} : f(x) = x$





The identity function

Proposition: The identity function is bijective.

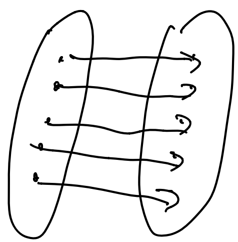
Proof: i_A is injective.

We must show that If $a, a' \in A$ and $i_A(a) = i_A(a')$ then $a = a'$.

Since $i_A(a) = a$ and $i_A(a') = a'$ clearly $a = a'$.

i_A is surjective.

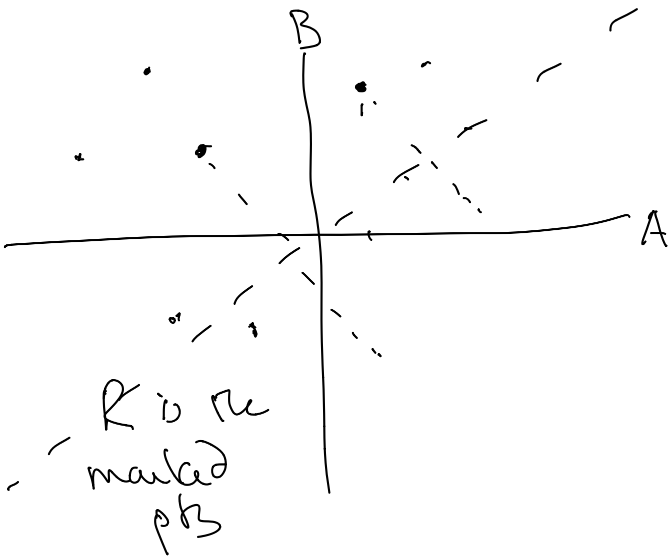
We must show that for any $a \in A$, there is an $a' \in A$ so that $i_A(a') = a$. Set $a' = a$, then $i_A(a') = i_A(a)$
$= a$.

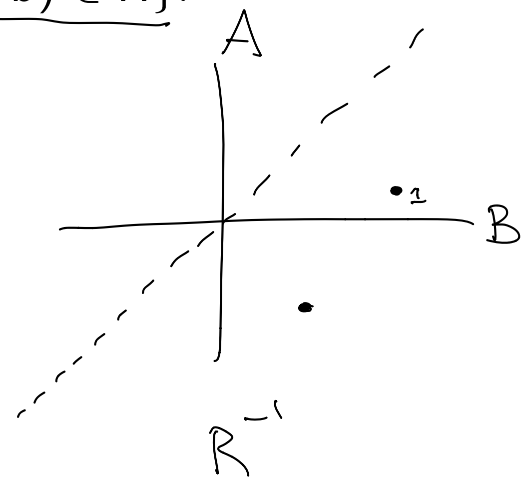


The inverse of a relation

Definition: Let A and B be sets and let R be a relation on $R \subset A \times B$. The inverse relation R^{-1} to R is the relation on $B \times A$ defined by

$$R^{-1} = \{(b, a) \in B \times A : \underbrace{(a, b) \in R}\}.$$



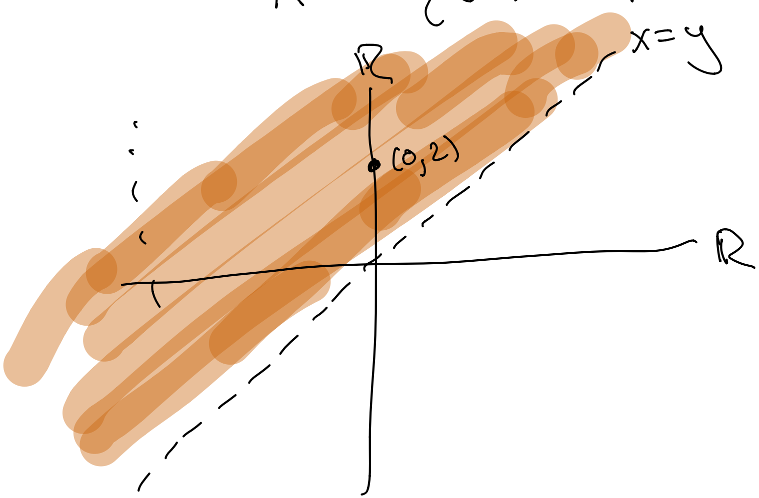


Examples of inverse relations

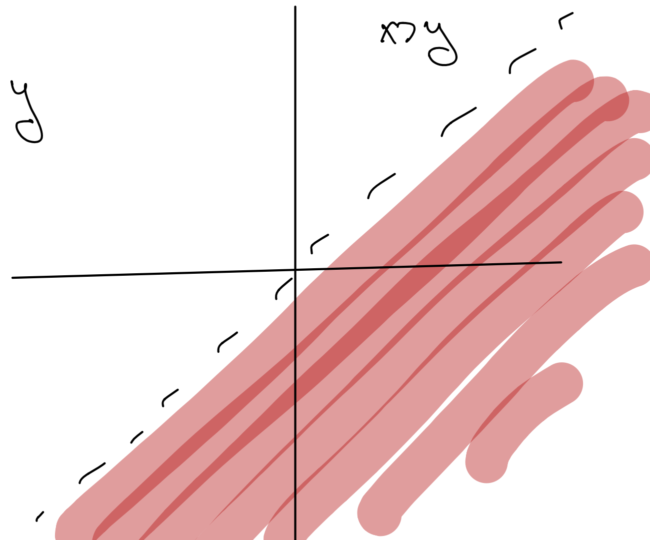
Example: Let $A = \mathbb{R}$ and let R be the relation $<$. Then R consists of all pairs $(a, b) \in \mathbb{R} \times \mathbb{R}$ with $a < b$. The inverse relation R^{-1} consists of all pairs $(b, a) \in \mathbb{R} \times \mathbb{R}$ with $(a, b) \in R$. Thus R^{-1} is the relation $>$.

$$R = \{(a,b) \mid a < b \quad a,b \in \mathbb{R}\}$$

$$R^{-1} = \{(b,a) \mid a < b \quad a,b \in \mathbb{R}\} = \{(a,b) \mid a > b\}$$



$R = <$



$R^{-1} = $ "$>$"

Another example

Example: Let $A = \mathbb{Z}$ and let R be the relation "divides", so that R consists of pairs $(a, b) \in \mathbb{Z} \times \mathbb{Z}$ where $a|b$. The inverse relation R^{-1} consists of pairs (a, b) where $b|a$, or, in other words, where a is a multiple of b.

So the inverse relation to $a|b$, meaning a is a divisor of b, is the relation aRb when a is a multiple of b.

R: $a|b$ if $\exists c \in \mathbb{Z}$ so that $b = ac$. $R \subseteq \mathbb{Z} \times \mathbb{Z}$

R^{-1}: $\{(b,a) \mid a|b,\ a,b \in \mathbb{Z}\} = \{(a,b) \mid b|a\}$

$aR^{-1}b$ means that a is a multiple of b