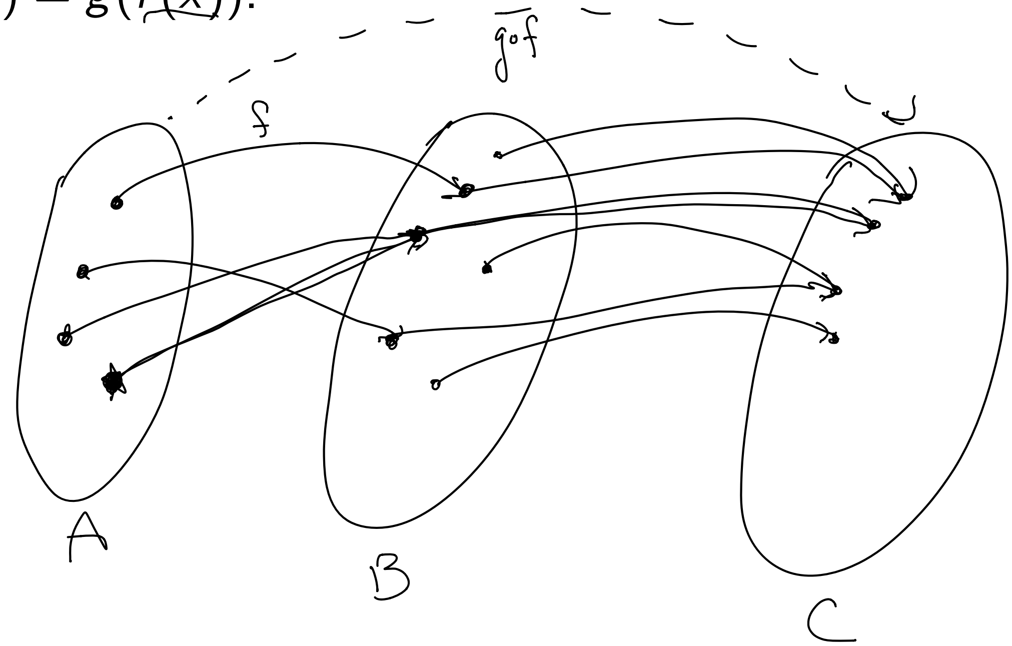
Composition of functions

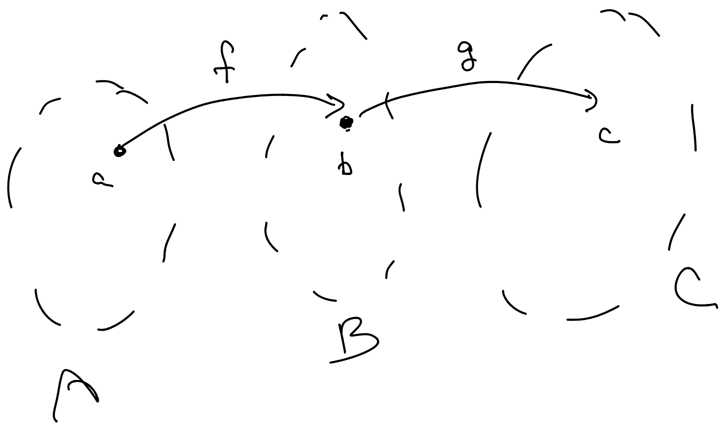
Suppose that $f : A \to B$ and $g : B \to C$ are functions. The *composition* of f and g is a new function $g \circ f : A \to C$ defined by $(g \circ f)(x) = g(f(x))$.



Composition cont'd

In terms of ordered pairs, if $\underbrace{f \subseteq A \times B}$ and $\underbrace{g \subseteq B \times C}$ are functions, then $g \circ f$ is the set of ordered pairs $(a, c) \in A \times C$ such that there exists $b \in B$ with $(a, b) \in f$ and $(b, c) \in g$.

$$g \circ f = \left\{ (a,c) \;\middle|\; a \in A,\ c \in C \text{ such that there is a } b \in B \text{ with } (a,b) \in f \text{ and } (b,c) \in g \right\}$$



Variations

- Suppose $f : A \to B$ and $g : C \to D$ are functions and $B \subseteq C$. Then we can still define $g \circ f$ by the same formula $(g \circ f)(x) = g(f(x))$.
- Suppose $f : A \to B$ and $g : C \to D$ are functions and the range of f is a subset of C. Then we can still define $(g \circ f)$ by the same formula.

$$\text{range}(f) = \{ b \in B \mid \exists a \in A, (a,b) \in f \}.$$

A warning

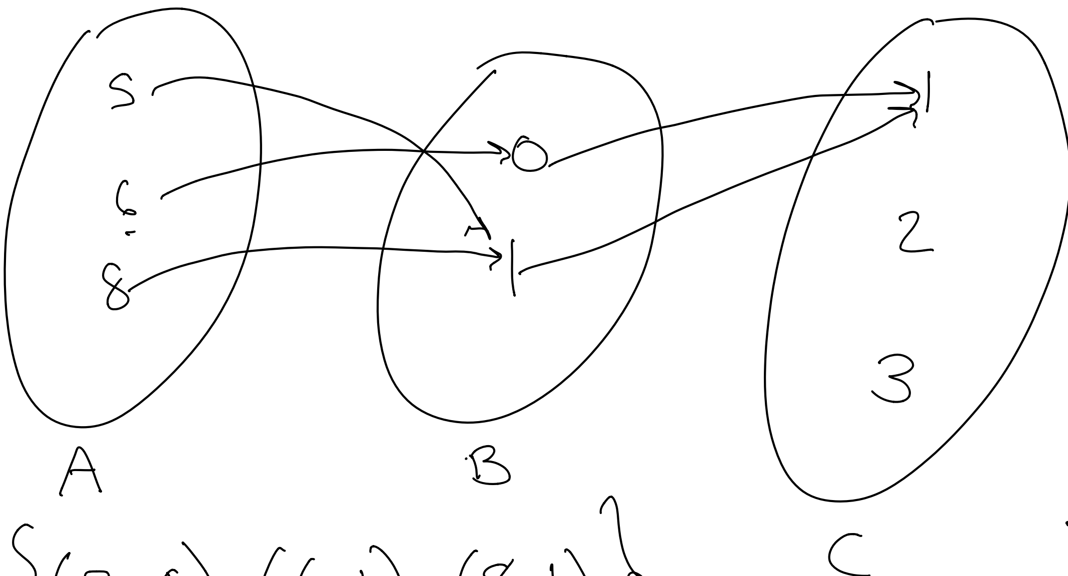
Warning: $g \circ f$ means *first* f, *then* g, NOT *first* g, *then* f, which is what our normal left-to-right instincts (at least in English) might suggest.

Examples

Problem 12.4.1: Suppose $A = \{5, 6, 8\}$, $B = \{0, 1\}$, and $C = \{1, 2, 3\}$. Let $f = \{(5, 1), (6, 0), (8, 1)\} \subseteq A \times B$ and let $g = \{(0, 1), (1, 1)\} \subseteq B \times C$. Find $g \circ f$.

First f

Then g



$(5,3) \notin g \circ f$

~~$(5,b)$~~ or

$(5,3) \in g \circ f$

$\Rightarrow \exists\, b \in B$

with $(5,b) \in f$

and $(b,3) \in g$.

FALSE, so $(5,3) \notin g \circ f$

$$g \circ f = \{(5,1), (6,1), (8,1)\}$$

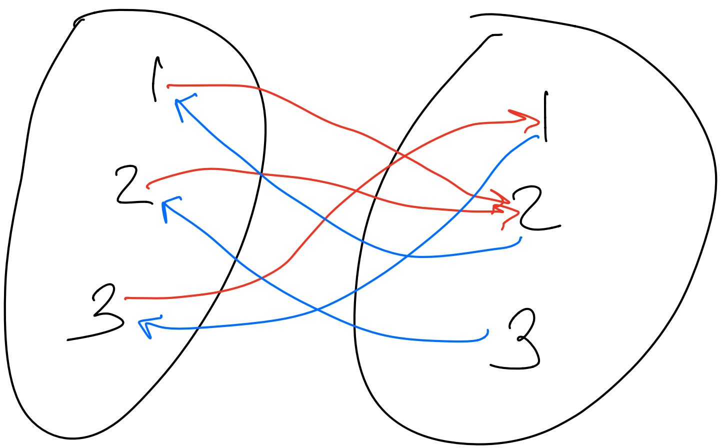
Examples continued

Problem 12.4.3: Let $A = \{1,2,3\}$ and let $f, g \subseteq A \times A$ be the function ~~$f = \{(1,3),(2,1),(3,2)\}$~~. Find $g \circ f$ and $f \circ g$.

$f = \{(1,2),(2,2),(3,1)\}$

$g = \{(1,3),(2,1),(3,2)\}$

Because $f: A \to A$, $g: A \to A$ it makes sense to look at $g \circ f$ and $f \circ g$



1) $g \circ f$: First f, then g.

$g \circ f: \{(1,1),(2,1),(3,3)\}$

2) $f \circ g$ First g, then f

$\{(1,1),(2,2),(3,2)\}$

Example that in general composition of functions isn't commutative.

Examples continued

$\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$

Problem 12.4.9: Let $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ be the function defined by $f(m, n) = m + n$ and $g : \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ be the function $g(m) = (m, m)$. Find the formulae for $g \circ f$ and $f \circ g$.

$g \circ f : \underline{\mathbb{Z} \times \mathbb{Z}} \to \mathbb{Z} \times \mathbb{Z}$

$f \circ g : \underline{\mathbb{Z}} \to \underline{\mathbb{Z}}$

$g \circ f\,(m,n) = g(f(m,n))$

$f(m,n) = m+n$

$g(m+n) = (m+n, m+n)$

$(g \circ f)(m,n) = (m+n, m+n)$

$(f \circ g)(m) = f(g(m))$

$= f(m,m) = m+m = 2m$

$(f \circ g)(m) = 2m.$

Proposition: Suppose that $f : A \to B$, $g : B \to C$ and $h : C \to D$ are functions. Then $(h \circ g) \circ f = h \circ (g \circ f)$. In other words, composition of functions is associative.

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$$

$$\underbrace{(h \circ g)}_{B \to D} \circ \underbrace{f}_{A \to B} \subseteq A \times D \qquad \cancel{(a, h\circ g(f(a)))}$$

$$= \{(a, h(g(f(a)))) \mid a \in A\}$$

$$\underbrace{h}_{C \to D} \circ \underbrace{(g \circ f)}_{A \to C} \subseteq A \times D =$$

$$(h \circ g)\circ f\,(a) = (h \circ g)(f(a)) = h(g(f(a)))$$

$$\| $$

$$h \circ (g \circ f)\,(a) = \cancel{h(g(f(a)}$$
$$h((g \circ f)(a)) = h(g(f(a)))$$

Theorem: Suppose $f : A \to B$ and $g : B \to C$ are functions.

- If f and g are injective, then $g \circ f$ is injective.
- If f and g are surjective, then $g \circ f$ is surjective.

Suppose f and g are injective. We will show that $g \circ f$ is injective. Suppose $(g \circ f)(x) = (g \circ f)(y)$. We will show that $x = y$.

$$(g \circ f)(x) = g(f(x))$$

$$(g \circ f)(y) = g(f(y))$$

so $g[f(x)] = g[f(y)]$

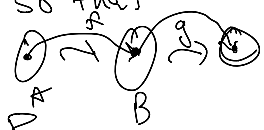
g injective. So $f(x) = f(y)$

f injective so $x = y$.

$\therefore$ $g \circ f$ is injective.

Suppose g and f are surjective.
We will show that $\cancel{g(f(}$ $(g\circ f)\cancel{(x)}$ is also surjective. $(g\circ f): A \to C$

Choose a $c\in C$. We must find $a\in A$ so that $(g\circ f)(a) = c$



Since g is surjective, there is a $b\in B$ so that $\cancel{g(a)\neq}$ $g(b) = c$.

Now since f is surjective there is an $a\in A$ so that $f(a) = b$.

Then $(g\circ f)(a) = g(f(a)) = g(b) = c$.

So $g\circ f$ is surjective.