Examples on Injective, Surjective, and Bijective functions

Example 12.4.

Proposition: The function $f : \mathbb{R} - \{0\} \to \mathbb{R}$ defined by the formula $f(x) = \frac{1}{x} + 1$ is injective but not surjective.

$f = \left\{ \left(x, \frac{1}{x} + 1\right) \text{ where } x \in \mathbb{R} - \{0\} \text{ and } y \in \mathbb{R} \right\}.$

Proof that f is injective.

Recall that f is injective if, for all $a, a' \in A$, if $f(a) = f(a')$ then $a = a'$.

So suppose $f(a) = f(a')$ then $\frac{1}{a} + 1 = \frac{1}{a'} + 1$

$\frac{1}{a} = \frac{1}{a'}$

$a = a'$

so f is injective.

Recall the f is surjective if, for all $b \in \mathbb{R}$, there is an $a \in \mathbb{R} - \{0\}$ so that $f(a) = b$.

$\frac{1}{a} + 1 = b$

$\frac{1}{a} = b - 1$

$a = \frac{1}{b-1}$ valid if $b \neq 1$.

If $b = 1$ then $\frac{1}{a} + 1 \neq 1$. f is not surjective because $b = 1$ is not in range of f.

$f(a) = \frac{1}{a} + 1$

Example 12.5.

Proposition: The function $f : \mathbb{R} - \{0\} \to \underline{\mathbb{R} - \{1\}}$ is injective and surjective (hence bijective).

Injectivity: if $\frac{1}{a} + 1 = \frac{1}{a'} + 1 \Rightarrow a = a'$

so f is injective.

Surjectivity. If $b \in \mathbb{R} - \{1\}$, we can solve

$$f(a) = \frac{1}{a} + 1 = b$$

$$a = \frac{1}{b-1}$$ which is valid since $b \neq 1$ since $1 \notin$ codomain of f.

Example 12.6

Proposition: The function $g : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ defined by the formula $g(m, n) = (m + n, m + 2n)$ is both injective and surjective.

$g(m,n) = (m+n, m+2n)$

Injectivity. Suppose $g(m,n) = g(m',n')$

so $(m+n, m+2n) = (m'+n', m'+2n')$

$$m+n = m'+n'$$
$$m+2n = m'+2n'$$

$$n = n'$$

Then $m+n = m'+n' = m'+n$

so $m = m'$.

$(m,n) = (m',n')$

therefore g is injective

Surjectivity. Given $(a,b) \in \mathbb{Z} \times \mathbb{Z}$, there are $(m,n) \in \mathbb{Z} \times \mathbb{Z}$

so that $g(m,n) = a,b$.

$$m+n = a$$
$$m+2n = b$$

$$n = b-a$$

$$m + b - a = a$$
$$m = 2a - b$$

$g(2a-b, b-a) = (a,b)$

Therefore g is surjective.

Example 12.15

Let A={A,B,C,D,E,F,G} and let $B = \{1, 2, 3, 4, 5, 6, 7\}$. How many functions are there from A to B? How many of these are injective? How many are surjective? How many are bijective?

COUNTING ALL FNS

$f: A \to B$

DOMAIN

CODOMAIN

A
B
C
D
E
F
G

For each of 7 elements of the domain we have 7 choices from codomain.

$7 \times 7 \times 7 \cdots 7 = 7^7$

COUNTING INJECTIVE FNS

A
B
C
D
E
F
G

7 choices for $f(A)$

6 choices for $f(B)$

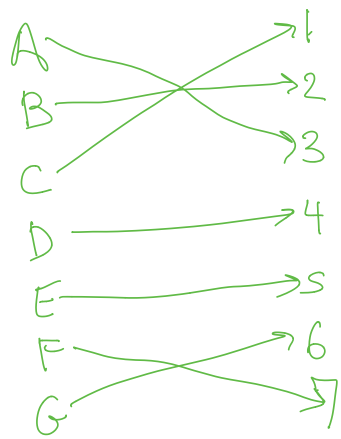
5 choices for $f(C)$

$\vdots$

1 choice for $f(G)$

$7!$

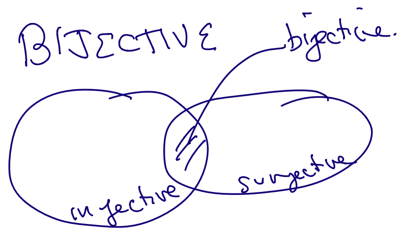
SURJECTIVE FUNCTIONS



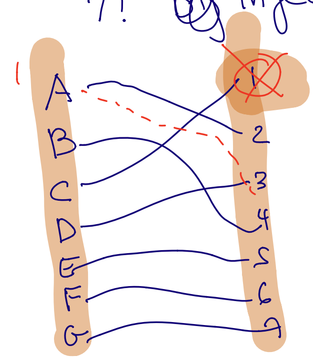
Every element in codomain has to be matched with atleast one element from the domain. But both sets have seven elements. So each element in codomain is matched with only one element in domain.

7! possibilites

BIJECTIVE



7! by injective functions ⇒ surjective.



7! bijective functions