Equivalence Relations and partitions

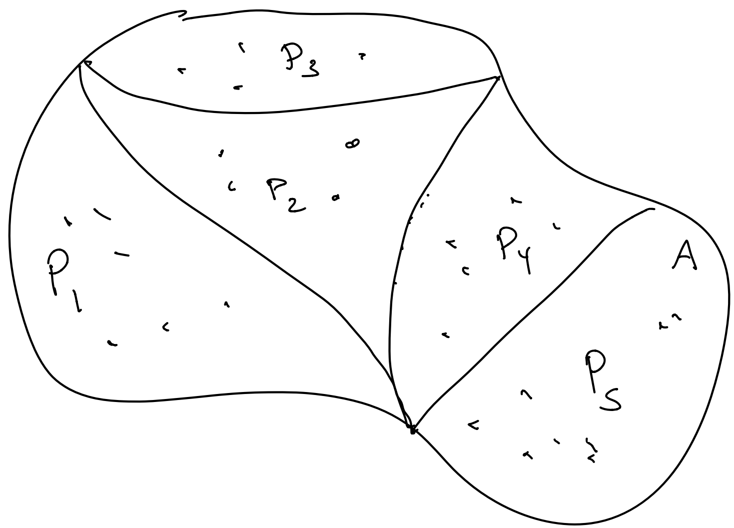
Partitions

Definition: A partition of a set A is a set of non-empty subsets of A such that the union of all of the subsets is A and the intersection of any two of the subsets is the empty set.

Intuitively: a partition is a division of A into disjoint subsets.

$P_i \cap P_j = \emptyset$ if $i \neq j$

$\bigcup_{i=1}^{5} P_i = A$



Partitions (Examples)

$\mathbb{Z} = $ Evens $\cup$ Odds

Even $\cap$ Odd $= \emptyset$

- Integers divided into even and odd integers. / Even Odd
- Integers divided into congruence classes modulo 3.
- Books with one author divided up into classes by author.
- People grouped by their county of residence.

3 congruence classes

$x \equiv 0 \mod 3 \quad [0] = \{0, 3, 6, 9, \ldots ; -3, -6, -9, \ldots\} = \{3k \mid k \in \mathbb{Z}\}$

$x \equiv 1 \mod 3 \quad [1] = \{-5, -2, 1, 4, 7, 10, \ldots\} = \{1+3k \mid k \in \mathbb{Z}\}$

$x \equiv 2 \mod 3 \quad [2] = \{\ldots, -4, -1, 2, 5, 8, \ldots\} = \{2+3k \mid k \in \mathbb{Z}\}$

Division algorithm: Every integer a can be written $a = 3q + r$ where $r = 0, 1, 2$ and this is unique.

Partitions and Equivalence Relations

Theorem (11.2): Let R be an equivalence relation on a set A. Then the equivalence classes $\{[a] : a \in A\}$ form a partition of A.

Proof: We have to show

1) $\bigcup_{a \in A} [a] = A$ $\quad$ $a \in [a]$ because aRa since R reflexive. ✓

2) if $[a] \neq [b]$ then $[a] \cap [b] = \emptyset$.

1) $[a] \subseteq A$ for all $a \in A$, $\bigcup_{a \in A} [a] \subseteq A$ $\quad$ Take $a \in A$. $[a] \subseteq \bigcup_{a \in A} [a]$ $\therefore A \subseteq \bigcup_{a \in A} [a]$

2) Suppose $[a] \cap [b] \neq \emptyset$. Then there is an $x \in [a]$ and $x \in [b]$.

Therefore xRa and xRb. By symmetry aRx

We have aRx and xRb so aRb. ~~So~~

$[a] = \{x \mid xRa\} = [b] = \{x \mid xRb\}$.

If $x \in [a]$ then xRa and aRb so xRb so $x \in [b]$

If $x \in [b]$ then xRb and aRb so bRa so xRa so $x \in [a]$

If $[a] \cap [b] \neq \emptyset$ then $[a] = [b]$. $\therefore$ if $[a] \neq [b]$ then $[a] \cap [b] = \emptyset$.

Converse to Theorem 11.2

Proposition: Suppose P is a partition of a set A. Define a relation R on A by setting aRb if and only if a and b belong to the same element of the partition. Then R is an equivalence relation.

As a result, partitions of a set are "the same" as equivalence relations on a set.

Define R

xRy if x, y belong to the same element of the partition.