Equivalence Relations

Equivalence Relation: Definition

$R \subseteq A \times A$

$aRb \Leftrightarrow (a,b) \in R$

Definition: Let A be a set. A relation R on A is called an *equivalence relation* if it is reflexive, symmetric, and transitive.

Reflexive: for all $a \in A$, $(a,a) \in R$ (aRa). ✓

Symmetric: For all $a, b \in A$, if $(a,b) \in R$ then $(b,a) \in R$ ✓

$aRb \Rightarrow bRa$.

Transitive: For all $a, b, c \in A$ ✓

if $(a,b) \in R$ and $(b,c) \in R$ then $(a,c) \in R$.

"$=$" is an equivalence relation.

$R = \{ (a,a) \mid a \in A \}$

Examples

Let $A = \{-2, -1, 1, 2, 3, 4\}$.

- The relation $=$ is an equivalence relation.
- The relation "has the same parity as" is an equivalence relation.
- The relation "has the same sign as" is an equivalence relation.
- The relation "has the same sign and parity" is an equivalence relation.

"same parity"

~~reflexive~~ ~~symmetric~~? yes a has same parity as a. does b have same as a? sure

symmetric: if a has same parity as b does b

transitive: if a and b have same parity and b and c do then a an c do.

sign: transitivy

a + b+ b+ c+ $\Rightarrow$ a+ and c+

a − b − b− c− $\Rightarrow$ a − and c −

Let X be the set of books in Babbidge Library with one author. Here are some equivalence relations:

- Has the same author.
- Has the same number of pages.
- Are located on the same floor of the library.

Equivalence Classes

n equivalence

Definition: Let A be a set and R a relation on A. For any $a \in A$, the *equivalence class* of a under R, written $[a]$ or $[a]_R$, is the set

$$[a]_R = \{b \in A : bRa\}.$$

$a \in [a]$

aRa true so $a \in [a]$.

If $A = \{-2, -1, 1, 2, 3, 4\}$ and R is the relation "has the same parity as" then:

- $[-2]$ is the set $\{-2, 2, 4\}$
- $[2]$ is the same set $\{-2, 2, 4\}$
- $[1]$ is the set $\{-1, 1, 3\}$
- $[3]$ is the set $\{-1, 1, 3\}$

$[-2] = [2]$

Equivalence Classes - Examples

If X is the set of books in Babbidge Library with one author, and R is the relation "has the same author" then

- [Ray Bradbury] is the set of books in Babbidge Library with only one author, and that author is Ray Bradbury.

If R is the relation "has the same number of pages", then

- [War and Peace] is the set of books in Babbidge Library (with one author) that have the same number of pages as War and Peace.

Question: why do I insist on books with one author?

R : aRb if the have a common author.

NOT AN EQUIV. RELATION

a : Techelbaum Frankenstein

b : Frankenstein Igor

c : Igor Kauffman

aRb

bRc

$a \not R c$

Example 11.12 - polynomials

Let P be the set of polynomials with real coefficients. Define a relation R on P by saying that fRg if f and g have the same degree. Then R is an equivalence relation.

The equivalence class $[x]$ of the polynomial x consists of all polynomials of degree one. $[x] = \{f : fRx\} = \{f : \deg f = \deg x\}$
$= \{f : \deg f = 1\}$.

More generally there is one equivalence class for each degree, and the equivalence class consists of all polynomials of that degree.

If f and g have degree n, and g and h have degree n,

$$\left.\begin{array}{l} f = ax^n + \dots \\ g = bx^n + \dots \\ h = cx^n + \dots \end{array}\right\}$$

then f and h have same degree.

Example 11.13 - Congruence

We have seen that $\equiv \pmod{n}$ is an equivalence relation on $\mathbb{Z}$.

What are the equivalence classes $[x]$ for $x \in \mathbb{Z}$?

$x \equiv y \mod n \iff n \mid (x-y) \Rightarrow x - y = kn$ for some k

$x = y + kn$ for some k.

$n = 3$,

$[1] = \{ x \mid x \equiv 1 \mod 3 \}$

$= \{ x \mid x = 1 + 3k, k \in \mathbb{Z} \}$

$[1] = \{ 1, 4, 7, 10, \ldots$
$-2, -5, -8, \ldots \}$

$[7]$

$\{ 7 + 3k \}$
$\|$
$\{ 1 + 6 + 3k \}$
$\|$
$\{ 1 + 3(k+2) \}$

$[2] = \{ -7, -4, -1, 2, 5, 8, \ldots \}$

$[0] = \{ -9, -6, -3, 0, 3, 6, 9, \ldots \}$

Rational numbers

Let M be the set of pairs (m, n) where m and n are integers and $n \neq 0$. Define a relation $(m, n)R(m', n')$ if $mn' - m'n = 0$. What are the equivalence classes?

(m, n) $\quad \underline{n \neq 0}, \quad m, n \in \mathbb{Z}$.

$(m,n) R (m', n')$ if $mn' - m'n = 0$

$[(m,n)] = \{ (m', n') \mid \underline{mn' - m'n = 0} \}$

$mn' = m'n$

$\frac{m}{n} = \frac{m'}{n'}$ $\quad n, n' \neq 0$ by definition

$[(1,2)] =$

$\{ (1,2), (2,4), (-2,-4), (-3,-6), \ldots$

$\frac{1}{2} = \frac{2}{4} \quad \frac{1}{2} = \frac{-2}{-4} \quad \frac{1}{2} = \frac{-3}{-6}$

$\equiv$ Continued.

$n \in \mathbb{N}$.

$[a] = \{ x \equiv a \bmod n \}$

$= \{ x = a + kn, \ k \in \mathbb{Z} \}$

$=$ arithmetic progression

$\{ a-3n, a-2n, a-n, a, a+n, a+2n, \ldots \}$

There are n different equivalence classes.

$[0], [1], \ldots\ldots, [n-1]$.