Properties of Relations

reflexivity

symmetry

transitivity

Reflexive Relations

$R \subseteq A \times A$

Definition: A relation R is **reflexive** if, for all $x \in A$, $(x, x) \in R$.
In other words, xRx for all $x \in A$.

$x = x$ true for all $x \in A$.

- The '$=$' relation is reflexive, as is the $\leq$ relation. $a \leq a$
- The $<$ relation is not reflexive. $a \not< a$ for all $a \in A$
- The "is an ancestor of" relation is not reflexive. $(P_1, P_2) \in R$ means P_1 is an ancestor of P_2
- The $\neq$ relation is not reflexive.

$x \neq x$ is false

Symmetric Relations

Definition: A relation R is **symmetric** if, for all $x, y \in A$, $xRy \implies yRx$. In other words, if $(x, y) \in R$ then $(y, x) \in R$.

$A = \mathbb{N}$

- The '$=$' relation is symmetric — if $x = y$ then $y = x$ true for all $x, y \in A$
- The $\leq$ relation is not symmetric — $\leq$ not symmetric: $\exists\, x, y$ so that $x \leq y$ and $y \not\leq x$. In particular $(3, 5)$: $3 \leq 5$ but $5 \not\leq 3$.
- The "is an ancestor of" relation is not symmetric.
- The $\neq$ relation is symmetric. — if $x \neq y$ then $y \neq x$.

If $A = \{1\}$, then $\leq$ is symmetric

$1 \leq 1$ $\qquad R = \{(1,1)\}$

$A = \emptyset \qquad R = \emptyset$.

Transitive relations

Definition: A relation R is **transitive** if, for all $x, y, z \in A$, if xRy and yRz then xRz. In other words, if $(x, y) \in R$ and (y, z) in R then $(x, z) \in R$.

$A = \mathbb{N}$

- The '$=$' relation is transitive if $a = b$ and $b = c$ then $a = c$
- The $\leq$ relation is transitive. if $a \leq b$ and $b \leq c$ then $a \leq c$
- The "is an ancestor of" relation is transitive.
- The $\neq$ relation is not transitive.

$3 \neq 5$ $\quad 5 \neq 3 \Rightarrow 3 \neq 3$ FALSE

$3, 5, 3$ shows $\neq$ is NOT transitive.

Example 11.7

Examine the properties reflexivity, symmetry, and transitivity when $A = \{b, c, d, e\}$ and

$$R = \{(b, b), (b, c), (c, b), (c, c), (d, d), (d, b), (b, d), (c, d), (d, c)\}$$

Reflexivity: $(a,a) \in R$ for all $a \in A$. (b,b), (c,c), (d,d), $\boxed{(e,e)}$ NOT

NOT Reflexive

Symmetry. 4 elements. ~~4 ⊗ 3~~ 16 ~~(a,c) (c,a)~~

if $(a,b) \in R$ then $(b,a) \in R$.

Transitivity $(b,c) \in R$ $(c,d) \in R$ if it were transitive then (b,d) would have to be in R. it is!

$(c,b) \in R$ $(b,d) \in R$ if transitive, $(c,d) \in R$ it is ✓

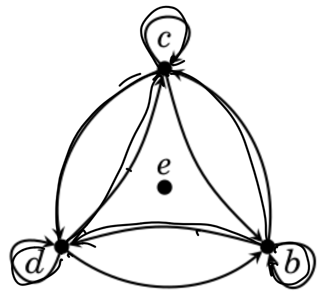
Example 11.7 continued

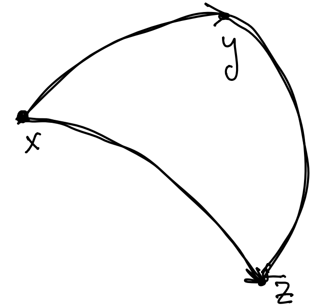
c d e

(c, d) in R yes

(d, e) in R NO

A picture





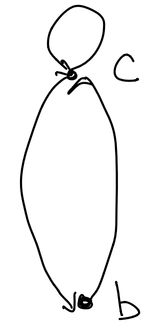


Figure 11.1. The relation R from Example 11.7

So everything is connected except e is isolated.

If xRy and yRz then xRz.

Congruence is reflexive, symmetric, and transitive.

Proposition: Let $\underbrace{n \in \mathbb{N}}$. The relation R on $\mathbb{Z}$ defined by aRb if and only if $\underbrace{a \equiv b \pmod{n}}_{n \mid (a-b).}$ is reflexive, symmetric, and transitive.

<u>Proof</u>:

1) Reflexivity. We must show that, for all $x \in \mathbb{Z}$, $x \equiv x \bmod n$. This is true because $x - x = 0$ and $n \mid 0$.

2) <u>Symmetry</u>: We must show that if $x \equiv y \bmod n$, then $y \equiv x \bmod n$.

<u>Proof</u>. If $n \mid (x-y)$, then $x - y = kn$.

so $(y-x) = (-k)n$ and so $n \mid (y-x)$

so $y \equiv x \bmod n$.

3) transitivity we must show that if $x \equiv y \mod n$ and $y \equiv z \mod n$ then $x \equiv z \mod n$.

Proof: $n \mid (x-y)$ so $x-y = kn$, for some k.

$n \mid (y-z)$ so $y-z = sn$ for some s

$$x - z = (k+s)n$$

so $n \mid (x-z)$ and $x \equiv z \mod n$.