Relations

Relations

Examples of Relations

- $=$, $<$, $>$, $\leq$, $\geq$, $\neq$, *etc.* are relations between numbers.

  $3 < 5$ $\pi \geq e$ $17 \neq 46$

- $\subseteq$ is a relation between sets

  $X \subseteq Y$

- "is the parent of" or "is a child of" or "is a spouse of" are relations between people.

- "comes earlier in the dictionary" is a relation between words.

Abstract Relations

Suppose we consider the relation $<$ on $\mathbb{N}$. We can "abstract" this relation by considering all pairs $(x, y) \in \mathbb{N} \times \mathbb{N}$ where $x < y$. Let R be the set of such pairs.

$R = \{(x, y) \mid x < y\}$

So $(1, 2) \in R$, but $(5, 4) \notin R$.

$(1,2) \in R$ ✓ $(5,4) \notin R$.

Once we have the set R, we know everything about $<$. Namely

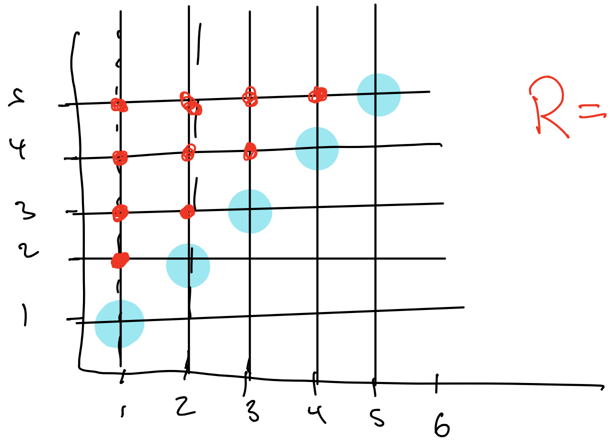
$$x < y \Leftrightarrow (x, y) \in R.$$

Now we *identify* the relation $<$ with this set R and we can study relations using set theory.

Pictures of relations

$x < y$

$R = \{(x, y) \mid x < y\}.$



R = all pts in $\mathbb{N} \times \mathbb{N}$ above the diagonal

A big picture

$A = \{\text{cities mostly in North America}\}$ $R = \{(x,y) \mid \text{there was}$

Here the underlying set is "North American Cities" and the relation is $(x, y) \in R$ if there was a United flight joining the two cities in 2019.

Abstract Relations: formal definition

Definition: Let A be a set. A *relation* on A is a subset R of the Cartesian product $A \times A$. We abbreviate the statement $(x, y) \in R$ as xRy, and $(x, y) \notin R$ as $x \not R y$.

Abstract relations: A few examples

- (Example 11.1) $A = \underline{\{1, 2, 3, 4\}}$ and R consists of

$$R = \underline{\{(1,1),(2,1),(2,2),(3,3),(3,2),(3,1),(4,4),(4,3),(4,2),(4,1)\}} \subseteq A \times A$$

$1R1$ $2R1$ $1\not R 2$ because $(1,2) \notin R$ [Secret: $(x,y) \in R \iff \underline{x \geq y}$].

- (Example 11.2) $A = \underline{\{1, 2, 3, 4\}}$ and S consists of

$$S = \{(1,1),(1,3),(3,1),(3,3),(2,2),(2,4),(4,2),(4,4)\} \subseteq A \times A$$

$1S1$ $2S2$ but $2 \not S 3$

$R \cap S = \{ (1,1)\ (3,1)\ (3,3)\ (2,2)\ (4,2)\ (4,4) \}$

$= \{ (x,y) \mid xRy$ and xSy
$x \geq y$ and x, y have same parity $\}$

Abstract Relations

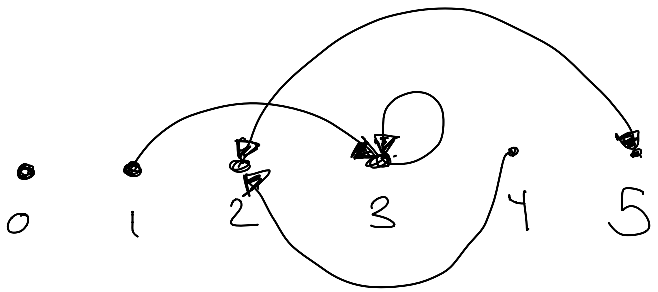
- (Example 11.3) The intersection of the two relations from the previous examples is a relation

$$\{(1,1),(2,2),(3,3),(3,1),(4,4),(4,2)\}$$

One more example

- (Example 11.4) $B = \{0, 1, 2, 3, 4, 5\}$ and

$$U = \{(1,3), (3,3), (5,2), (2,5), (4,2)\} \subseteq B \times B.$$

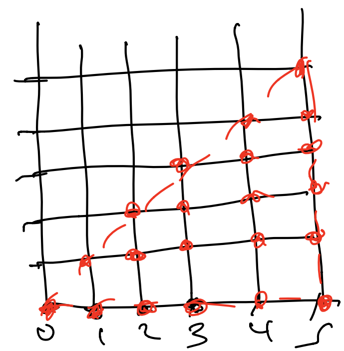


Problem 3, page 204.

- Let $A = \{0, 1, 2, 3, 4, 5\}$. Write out the relation R that expresses $\geq$ on A and illustrate it with a diagram.

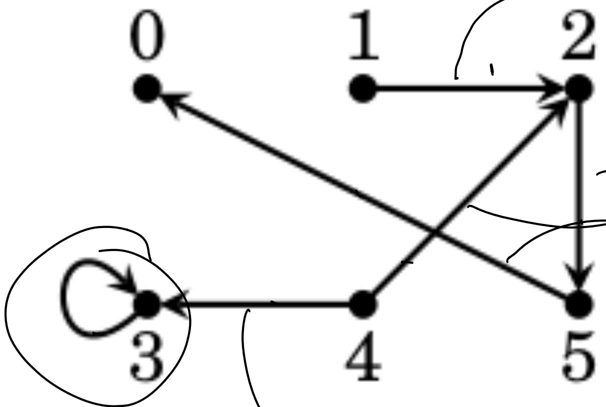
$\geqslant$ $\qquad R = \{(x, y) \mid x \geq y\}$.

$$\left\{\begin{array}{cccccc} (0,0) & (1,0) & (2,0) & (3,0) & (4,0) & (5,0) \\ & (1,1) & (2,1) & (3,1) & (4,1) & (5,1) \\ & & (2,2) & (3,2) & (4,2) & (5,2) \\ & & & (3,3) & (4,3) & (5,3) \\ & & & & (4,4) & (5,4) \\ & & & & & (5,5) \end{array}\right\} = R$$



Problem 5, page 204.

Write out the sets A and $R \subseteq A \times A$ described by this diagram.



(1, 2)
(2, 5)
(4, 2)
(4, 3)
(3, 3)
(5, 0)