Fibonacci numbers

Fibonacci Numbers

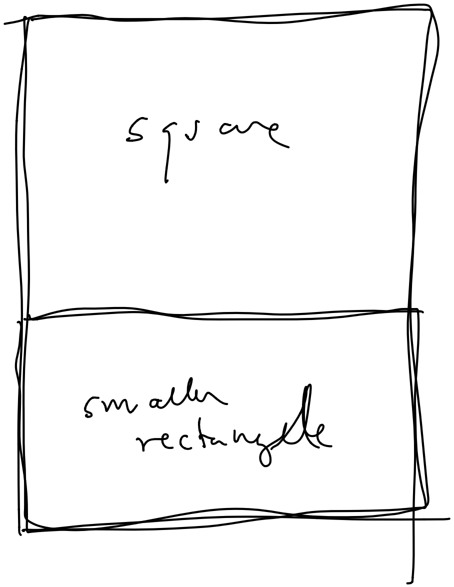
Figure 1: Fibonacci

The Fibonacci numbers F_n are defined by a *recursive* formula. The first two numbers are given by $F_1 = 1$ and $F_2 = 1$ and, for all $n \geq 3$, $F_n = F_{n-1} + F_{n-2}$.

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$$

Fibonacci Numbers and the Golden Ratio

See Donald Duck in Mathmagic Land (7 minute mark - 14 minute mark).



Fibonacci Numbers and the Golden Ratio

The golden ratio

$$\phi = \frac{1+\sqrt{5}}{2}$$

is the larger root of the quadratic polynomial $x^2 - x - 1 = 0$.

Proposition: The ratio of successive Fibonacci numbers F_{n+1}/F_n converges to the Golden ratio.

1, 1, 2, 3, 5, 8, 13, 21, 34, - - - - -

$1/1$, $2/1$, $3/2$, $5/3$, $8/5$, - - - $F_{n+1}/F_n \longrightarrow \phi$

Some Data

1	1	1.000000000
1	2	2.000000000
2	3	1.500000000
3	5	1.666666667
5	8	1.600000000
8	13	1.625000000
13	21	1.615384615
21	34	1.619047619
34	55	1.617647059
55	89	1.618181818
89	144	1.617977528
144	233	1.618055556
233	377	1.618025751
377	610	1.618037135
610	987	1.618032787

$\approx \frac{1+\sqrt{5}}{2}$

Fibonacci Numbers cont'd

Proposition: $F_{n+1}^2 - F_n F_{n+1} - F_n^2 = (-1)^n$.

$$3^2 - (2)(3) - 2^2 = -1 \qquad 3, 2$$

$$5^2 - (3)(5) - 3^2 = +1 \qquad 5, 3$$

$$8^2 - (5)(8) - 5^2 = -1 \qquad 8, 5$$

Corollary: $\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \phi$.

Proof: Divide through by F_n^2:

$$T^2 - T - 1 = 0$$

$$F_n \to \infty$$

$$\left(\frac{F_{n+1}}{F_n}\right)^2 - \left(\frac{F_{n+1}}{F_n}\right) - 1 = \frac{(-1)^n}{F_n^2} \to 0$$

The right hand side goes to zero, so (F_{n+1}/F_n) converges to a root of the polynomial which is greater than one.

Proof of proposition

$F_2 = 1$
$F_1 = 1$

First check that $F_2^2 - F_1F_2 - F_1^2 = -1$, which is $1^2 - 1 - 1 = -1$ as we want.

$1 - 1 - 1 = -1 = (-1)^n$

- Now suppose that the formula holds for F_n, so $F_n^2 - F_nF_{n-1} - F_{n-1}^2 = (-1)^{n-1}$.

  F_n, F_{n-1}

- Consider $F_{n+1}^2 - F_{n+1}F_n - F_n^2$.
- Substitute $F_{n+1} = F_n + F_{n-1}$ to get

$$\overbrace{(F_n + F_{n-1})^2}^{F_{n+1}^2} - \overbrace{(F_n + F_{n-1})}^{F_{n+1}}\overbrace{F_n}^{F_n} - \overbrace{F_n^2}^{F_n^2} =$$

$$F_n^2 + 2F_nF_{n-1} + F_{n-1}^2 - F_n^2 - F_{n-1}F_n - F_n^2$$

Then the right hand side of this equation is

$$-F_n^2 + F_nF_{n-1} + F_{n-1}^2 = -\overbrace{(F_n^2 - F_nF_{n-1} - F_{n-1}^2)}\underset{(-1)^{n-1}}{} = (-1)^n$$

where we used the inductive hypothesis to in the second-to-last step.