Strong Induction Continued

Exercise 13, page 195

Proposition: Prove that $6|(n^3 - n)$ for all $n \geq 1$.

Dry Run. $P(1)$: $6 | 1^3 - 1$ or $6|0$ TRUE

$P(n)$: $6 | n^3 - n$

Suppose $P(n)$ is true, so $6 | n^3 - n$.

Is $P(n+1)$ true? Does $6 | (n+1)^3 - (n+1)$

$$(n+1)^3 = n^3 + 3n^2 + 3n + 1$$

$$n+1 \qquad\qquad n+1$$

$$(n+1)^3 - (n+1): \quad n^3 + 3n^2 + 2n$$

doesn't look divisible by 6!

Try again with strong induction.

$P(2)$: $6 | 2^3 - 2 = 6|6$ TRUE

$P(3)$: $6 | \cancel{8}3^3 - 3 = 6|(27-3) = 6|24$ TRUE

$P(n+2)$: $(n+2)^3 - (n+2) = n^3 + 6n^2 + 12n + 8$

$$n + 2$$

$$n^3 + 6n^2 + 12n + 6 - n$$

$$= \underline{(n^3 - n)} + (6n^2 + 12n + 6)$$

$(n^3 - n)$ is divisible by 6 so $P(n+2)$ is true

if $P(n)$ is true.

$P(n) \Rightarrow P(n+2)$ for all n.

$P(\underline{n-1}) \Rightarrow P(n+1)$ for all n.

If $P(1)$ is true and

$$P(1) \wedge \dots \wedge P(n) \Rightarrow P(n+1)$$

then $P(n)$ is true for all n.

We can apply this to our situation and get that $6 \mid n^3 - n$ for all n.

Another Look

$$P(1) \Rightarrow P(3) \Rightarrow P(5) \Rightarrow P(7) \dots$$

$P(n)$ is true when n is odd.

If n odd, then $6 \mid n^3 - n$.

$$P(2) \Rightarrow P(4) \Rightarrow P(6) \Rightarrow \dots$$

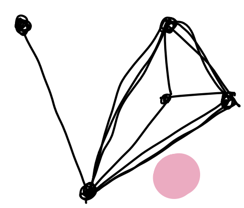
If n is even then $6 \mid n^3 - n$

$6 \mid n^3 - n$ for all n.

Trees (text, pages 189-190)

Proposition: A tree with n vertices has $n-1$ edges.

graph

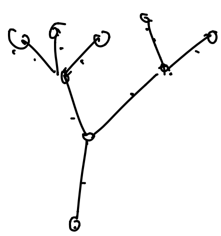


a graph

set of vertices V

set of edges $E \subseteq V \times V$

"tree" is a graph w/o cycles.



9 vertices

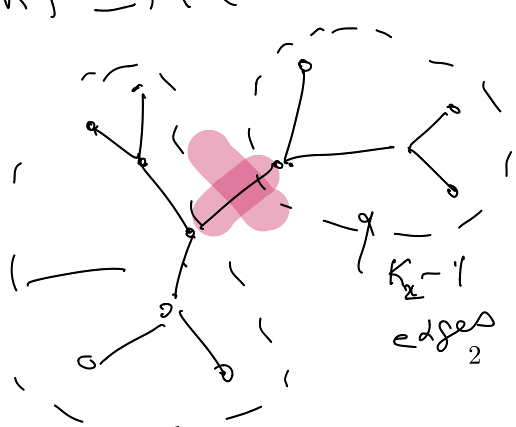
8 edges

Proof: If tree has one vertex then zero edges

$P(n)$: A tree with n vertices has $n-1$ edges

Prove $P(n) \Rightarrow P(n+1)$. Take a tree with $n+1$ vertices. Pick an edge and remove it from tree. Now you have two trees. Each have fewer than n vertices. tree 1 has K_1 vertices and $K_1 - 1$ edges

$K_1 - 1$ edges

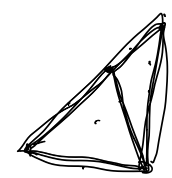


$K_2 - 1$ edges$_2$

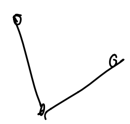
tree 2 has K_2 vertices
and $K_2 - 1$ edges

Combining:
original tree has $n+1 = K_1 + K_2$ vertices
$(K_1 - 1 + K_2 - 1) + 1$ edges.
$n_1 + K_2 - 1 = n$ edges.

This proves the proposition



Fact: Connected graph



$$V - E = 1 - \#\text{"cycles"}$$

$$\#\text{cycles} = 1 - V + E$$

$$V = E + 1$$

$$1 - (1 + E) + E = 0$$