More examples

Problem 9: Prove that $24|(5^{2n}-1)$ for all integers $n \geq 0$.

$n=0$: $5^{2n}-1=0$ and $24|0$.

$n=1$: $5^{2n}-1=24$ and $24|24$

Goal is to show that

Inductive Hypothesis: if $24|5^{2n}-1$ then $24|5^{2(n+1)}-1$. $(*)$

$$5^{2n}-1=(5^2)^n-1 = 25^n-1$$
$$= (1+24)^n-1 =: A$$
$$5^{2(n+1)}-1=(5^2)^{(n+1)}-1 = 25^{(n+1)}-1$$
$$= (1+24)^{n+1}-1 =: B$$

$$(1+24)(A+1)-1$$
$$=(1+24)(1+24)^n-1 = (1+24)^{n+1}-1 = B$$

$$1(A+1)-1+24(A+1) = B$$
$$A+1-1+24(A+1)=B$$
$$B = A+24(A+1)$$

We know that $24|A$, so $A=24K$ for some K.

$$B=24K+24(A+1) = 24(K+A+1)$$

B is a multiple of 24,

this proves $(*)$, so by induction $5^{2n}-1$ is always a multiple of 24. $(n\geq 0)$

Problem 21: Prove that

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2^n-1}+\frac{1}{2^n}\geq 1+n/2 \qquad P(n)$$

and conclude that the harmonic series diverges.

Check $P(1)$:

$n=1$ $\quad 1+\frac{1}{2} \geqslant 1+1/2$ $\quad$ true $\quad$ $P(1)$ holds.

$n=2$ $\quad 1+\frac{1}{2}+\underbrace{\frac{1}{3}+\frac{1}{4}}_{\text{2 terms}} \geqslant 1+\frac{1}{2}+2\left(\frac{1}{4}\right)=1+\frac{1}{2}+\frac{1}{2}$

$= 1+\frac{2}{2}$

all bigger than or equal to the last term $\frac{1}{4}$

Assume now that

$$1+\frac{1}{2}+\frac{1}{3}+\cdots\cdots+\frac{1}{2^n} \geqslant 1+n/2,$$

Add

$$1+\frac{1}{2}+\cdots\cdots+\frac{1}{2^n}+\underbrace{\boxed{\frac{1}{2^n+1}+\frac{1}{2^n+2}+\cdots\cdots+\frac{1}{2^{n+1}}}}_{2^n \text{ Terms}}$$

2^n terms all $\geqslant \frac{1}{2^{n+1}}$

$$\underbrace{\underbrace{1+\frac{1}{2}+\cdots+\frac{1}{2^n}}+\frac{1}{2^n+1}+\cdots+\frac{1}{2^{n+1}}} \geqslant \underbrace{1+\cdots+\frac{1}{2^n}}+\underbrace{2^n\cdot\frac{1}{2^{n+1}}} \geqslant 1+n/2+\frac{1}{2}$$

$$'' \quad \underline{1+\frac{(n+1)}{2}}$$

We've shown $P(n) \Rightarrow P(n+1)$

So by induction $P(n)$ is true for all $n \geqslant 0$ in $\mathbb{Z}$.

Corollary: Harmonic series diverges.