Induction, continued

Induction

In Section 10.1, the book proves the following propositions by applying the axiom of induction.

1. If $n \in \mathbb{N}$, then $1 + 3 + 5 + \cdots + (2n-1) = n^2$
2. If n is a non-negative integer, then $5|(n^5 - n)$.
3. If $n \in \mathbb{Z}$, and $n \geq 0$, then $\sum_{i=0}^{n} i \cdot i! = (n+1)! - 1$.
4. If $n \in \mathbb{N}$, then $2^n \leq 2^{n+1} - 2^{n-1} - 1$.
5. If $n \in \mathbb{N}$, then $(1+x)^n \geq 1 + nx$ for all $x \in \mathbb{R}$ with $x > -1$.

YOU SHOULD CAREFULLY STUDY ALL OF THESE PROOFS

Two notes: Problem 3 has $n \geq 0$ and Problem 5 has an additional variable.

$P(1)$, $P(2)$, $P(3)$

Induction axiom: Given a family of statements $P(n)$ for $n \in \mathbb{N}$, if $P(1)$ is true and, for all $n \in \mathbb{N}$, $P(n) \Rightarrow P(n+1)$, then $P(n)$ is true for all $n \in \mathbb{N}$.

$Q(0)$, $Q(1)$, $Q(2)$, $\vdots$

Proposition ~~Corollary~~: Suppose $Q(n)$, $n \in \mathbb{Z}, n \geq 0$ is a family of statements. If $Q(0)$ is true and $Q(n) \Rightarrow Q(n+1)$ for all $n \in \mathbb{Z}$, $n \geq 0$, then all $Q(n)$ are true.

Proof. Define $P(n) = Q(n-1)$ for $n \geq 1$ $(n \in \mathbb{N})$.

$P(1)$ is $Q(0)$

$P(2)$ is $Q(1)$

By hypothesis, $Q(0)$ is true so $P(1)$ is true.

for $n \geq 0$, $Q(n) \Rightarrow Q(n+1)$ so $P(n+1) \Rightarrow P(n+2)$ starting at $n = 0$

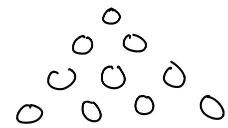
so $P(1) \Rightarrow P(2)$ and so on.

By axiom of induction, $P(n)$ is true for $n = 1, 2, \ldots$

$\therefore$ $Q(n) = P(n+1)$ is true for $n = 0, 1, 2, \ldots$

Triangular numbers (Exercise 1)

Proposition: Prove that $1 + 2 + 3 + \cdots + n = \frac{n^2+n}{2}$.



$1 + 2 + 3 + 4 = 10$

$1+2+3+4 = \frac{4^2+4}{2} = \frac{20}{2} = 10$

$P(1): \quad 1 = \frac{1^2+1}{2} = \frac{2}{2} = 1$ ✓ true

$P(2): \quad 1+2 = \frac{2^2+2}{2} = \frac{6}{2} = 3$ ✓ true

To apply the axiom of induction we must show that

$P(n) \Rightarrow P(n+1)$ for all $n \in \mathbb{N}$.

☆ $P(n): \quad 1+2+ \cdots + n = \frac{n^2+n}{2}$

☆☆ $P(n+1): \quad 1+2+ \cdots + (n) + n+1 = \frac{(n+1)^2 + (n+1)}{2}$

Assume ☆. must show ☆☆ follows.

$1+2+ \cdots + n = \frac{n^2+n}{2}$

$1+2+ \cdots + n + n+1 = \frac{n^2+n}{2} + n+1 = \frac{n^2+n+2n+2}{2}$

$= (n^2+3n+2)/2$

$= \frac{(n+1)^2+(n+1)}{2}$

$P(n+1)$ is a consequence of $P(n)$ so the formula holds for all $n \in \mathbb{N}$

Geometric series

Proposition: For any $n \geq 0$,

$$1 + x + \cdots + x^n = \frac{x^{n+1} - 1}{x - 1}$$

$n = 1$

$1 + x = \frac{x^2 - 1}{x - 1}$

$n = 2$

$1 + x + x^2 = \frac{x^3 - 1}{x - 1}$

$x^3 - 1 = (x-1)(x^2 + x + 1)$

$P(1)$: $1 + x = \frac{x^2 - 1}{x - 1}$ TRUE

$P(n)$: $1 + x + \cdots + x^n = \frac{x^{n+1} - 1}{x - 1}$

$P(n+1)$: $1 + x + \cdots + x^{n+1} = \frac{x^{n+2} - 1}{x - 1}$

Must show that $P(n) \Rightarrow P(n+1)$.

Assume $1 + x + \cdots + x^n = \frac{x^{n+1} - 1}{x - 1}$ $P(n)$

Show $P(n+1)$

$$1 + x + \cdots + x^n + x^{n+1} = \frac{x^{n+1} - 1}{x - 1} + x^{n+1}$$

$$= \frac{x^{n+1} - 1 + x^{n+1}(x-1)}{x - 1}$$

$$= \frac{x^{n+1} - 1 + x^{n+2} - x^{n+1}}{x - 1}$$

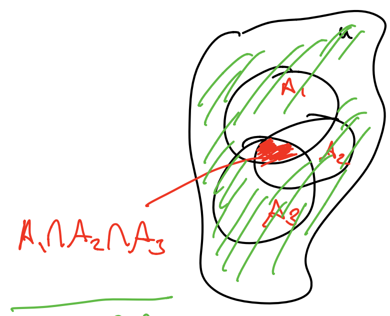
$$= \frac{x^{n+2} - 1}{x - 1}$$

so formula holds for all $n \in \mathbb{N}$ by induction.

A result on sets (Problem 17)

Proposition: Suppose that $A_1, A_2, \cdots, A_n$ are sets contained in a universal set U and that $n \geq 2$. Then

$$\overline{\bigcap_{i=1}^{n} A_i} = \bigcup_{i=1}^{n} \overline{A_i}$$



$$\overline{A_1} \cup \overline{A_2} \cup \overline{A_3} = \overline{A_1 \cap A_2 \cap A_3}$$

$P(2): \quad \overline{A_1 \cap A_2} = \overline{A_1} \cup \overline{A_2}$

$P(3): \quad \overline{A_1 \cap A_2 \cap A_3} = \overline{A_1} \cup \overline{A_2} \cup \overline{A_3}$

$\vdots$

is $P(2)$ true? is $\overline{A_1 \cap A_2} = \overline{A_1} \cup \overline{A_2}$

$x \in \overline{A_1 \cap A_2} \iff x \notin A_1 \cap A_2$

$\iff$ either $x \notin A_1$ or $x \notin A_2$

$\iff$ $\cancel{x \notin A_1}$ $x \in \overline{A_1} \cup \overline{A_2}$

Assume

$\overline{A_1 \cap A_2 \cap \cdots \cap A_n} = \overline{A_1} \cup \overline{A_2} \cdots \cup \overline{A_n}$. $\qquad P(n)$

Show $\overline{A_1 \cap A_2 \cap \cdots \cap A_n \cap A_{n+1}} = \overline{A_1} \cup \overline{A_2} \cdots \cup \overline{A_n} \cup \overline{A_{n+1}}$ $\qquad P(n+1)$

Let $X = A_1 \cap A_2 \cap \cdots \cap A_n$

$\xrightarrow[P(n+1)]{} \overline{A_1 \cap \cdots \cap A_n \cap A_{n+1}} = \overline{X \cap A_{n+1}} = \overline{X} \cup \overline{A_{n+1}} = \overline{A_1} \cup \cdots \cup \overline{A_n} \cup \overline{A_{n+1}}$

$\overline{X} = \overline{A_1} \cup \cdots \cup \overline{A_n}$ by $P(n)$