Introduction to mathematical induction

A sample problem

Proposition: The sum of the first n odd natural numbers is n^2.

The first n odd numbers are $\underline{1, 3, \ldots, 2n-1}$. So in summation notation this is the claim that, for all $n \in \mathbb{N}$,

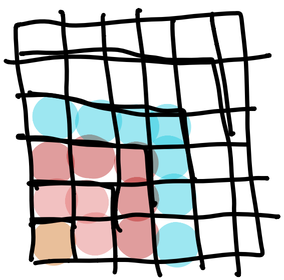
$$\sum_{i=1}^{n} \underline{(2i-1)} = n^2$$

.

$2i+1$ is odd as $i \in \mathbb{N}$

$i=1$ 1st odd number which is 1

$n=1$
$1=1^2$
$n=2$
$1+3=4=2^2$
$n=3$
$1+3+5=9=3^2$



Sample problem continued.

The proposition above is infinitely many statements.

$P \Rightarrow Q$

$P \Rightarrow P_1 \Rightarrow P_2 \Rightarrow P_3 \Rightarrow \cdots$ Q

n	sum of the first n odd natural numbers	n^2
1	$1 =$	1
2	$1+3 =$	4
3	$1+3+5 =$	9
4	$1+3+5+7 =$	16
5	$1+3+5+7+9 =$	25
$\vdots$	$\vdots$	$\vdots$
n	$1+3+5+7+9+11+\cdots+(2n-1) =$	n^2
$\vdots$	$\vdots$	$\vdots$

Figure 1: From pg. 180 of the text

Sample continued

We can prove any *one* of these statements.

How do we prove *all* of them?

Mathematical induction

Mathematical induction *extends* our system of logic by adding an axiom.

Axiom of Induction: Let $P(n)$ be a collection of statements, one for each natural number. Suppose that $P(1)$ is true and, for all n, the implication $P(n) \implies P(n+1)$ is true. Then $P(n)$ is true for all n.

The book calls this a *method of proof* but it is really an axiom.

$P(1)$ true

Want to be true $[P(1) \Rightarrow P(2) \Rightarrow P(3) \Rightarrow \cdots\cdots]$

$P(1) \Rightarrow P(2)$

$P(2) \Rightarrow P(3)$

$P(3) \Rightarrow P(4)$

$\vdots$

A prototype

Proposition: Suppose that S is a set such that $1 \in S$ and, for all $n, \in \mathbb{N},$ if $n \in S$, then also $n+1 \in S$. Then $\mathbb{N} \subseteq S$.

Proof: Let $P(n)$ be the statement $n \in S$. The hypotheses say that $P(1)$ is true, and that $P(n) \implies P(n+1)$. Therefore $P(n)$ is true for all n, and so every natural number is in S, so $\mathbb{N} \subset S$.

$P(1)$: $1 \in S$

$P(2)$ $2 \in S$

$P(3)$. $3 \in S$

$\vdots$

Given: $n \in S \Rightarrow n+1 \in S$.

By induction: since

$P(1)$ true $(1 \in S)$

$P(n) \Rightarrow P(n+1)$

if $n \in S$ then $n+1 \in S$

Conclude $P(n)$ is true for all $n \in \mathbb{N}$

$n \in S \ \forall n \in \mathbb{N}$.

so $\mathbb{N} \subseteq S$.

Proof of the result on sum of odd numbers

Proposition: For all n, we have

$$\sum_{i=1}^{n}(2n-1) = n^2.$$

Proof: We apply mathematical induction. The statement $P(n)$ is

$$P(n): \quad \sum_{i=1}^{n}(2n-1) = n^2.$$

So $P(1)$ is the claim that $1 = 1^2$, which is true. To prove that $\underline{P(n) \implies P(n+1)}$, we assume $P(n)$ true:

$$P(n): \quad 1 + 3 + 5 + \cdots + (2n-1) = n^2.$$

$$P(n+1): \quad 1 + 3 + 5 + \cdots + (2n-1) + \cancel{(2n+1)} = (n+1)^2$$

$$2(n+1) - 1$$

Proof, continued

$P(n) \Rightarrow P(n+1)$.

$$1+3+5+\cdots+(2n-1)+(2(n+1)-1) =$$
$$\boxed{1+3+5+\cdots+(2n-1)}+2n+1 =$$
$$n^2+2n+1 = (n+1)^2.$$

Therefore, if $P(n)$ is true then $P(n+1)$ is also true. By mathematical induction $P(n)$ is true for all n.