Indexed sets

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Summation notation

"Recall" that we can write a long sum of a bunch of numbers using summation notation.

$$\overbrace{a_1 + a_2 + \cdots + a_n}^{n \text{ number}} = \sum_{i=1}^{n} a_i$$

$a_1 + a_2 + a_3 + \cdots + a_n$

$$4 + \underbrace{7 + 8 + 13 + 4}_{n=5}.$$

We can even write infinite sums:

$$1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^i} + \cdots = \sum_{i=1}^{\infty} \frac{1}{2^i}$$

i takes on every value in $\mathbb{N}$

Indexed sets

Suppose we have a bunch of sets $\underbrace{A_1, A_2, \ldots, A_n}_{\text{all sets}}$. Then we can write:

$$A_1 \cup A_2 \cup \cdots \cup A_n = \bigcup_{i=1}^{n} A_i \qquad \bigcup_{i=1}^{n} \underline{A_i}$$

and

$$A_1 \cap A_2 \cap \cdots \cap A_n = \bigcap_{i=1}^{n} A_i$$

Indexed sets

If $A_1, A_2, \ldots, A_n$ are all sets, then

$$\bigcup_{i=1}^{n} A_i = \{x : x \text{ belongs to at least one set } A_i\}$$

- $A_1 = \{1, 4, 10, 12\}$
- $A_2 = \{5, 12, 15\}$
- $A_3 = \{1, 4, 15, 35\}$

$$Y = \bigcup_{i=1}^{3} A_i = A_1 \cup A_2 \cup A_3$$

$$= \{x : x \in A_1 \text{ or } x \in A_2 \text{ or } x \in A_3\}$$

What is $\bigcup_{i=1}^{3} A_i$?

$$Y = \{1, 4, 10, 12, 5, 15, 35\}$$

Indexed sets

$$\bigcap_{i=1}^{n} A_i = \{x : x \text{ belongs to every set } A_i\}$$

- $A_1 = \{\cancel{1}, \cancel{4}, \cancel{10}, \cancel{12}\}$
- $A_2 = \{\cancel{5}, \cancel{12}, \cancel{15}\}$
- $A_3 = \{\cancel{1}, \cancel{4}, \cancel{15}, \cancel{35}\}$

What is $\bigcap_{i=1}^{3} A_i$?

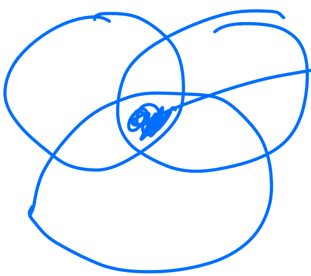
$$Y = \bigcap_{i=1}^{3} A_i = A_1 \cap A_2 \cap A_3$$

$1 \notin A_2$ $\quad$ $5 \notin A_1$

$4 \notin A_2$ $\quad$ $15 \notin A_1$

$10 \notin A_2$ $\quad$ $35 \notin A_2$

$12 \notin A_3$



Nothing here

$$Y = \emptyset$$

Indexed sets

One can also take the union and intersection of infinitely many sets.

$\bigcup_{i=1}^{\infty} A_i$ and $\bigcap_{i=1}^{\infty} A_i$.

$A_1 = \{-1, 0, 1\} \subseteq \mathbb{Z}$

Example. For each $i \in \mathbb{N}$, let

$A_2 = \{-2, 0, 2\} \subseteq \mathbb{Z}$

$A_3 = \{-3, 0, 3\} \subseteq \mathbb{Z}$

$$A_i = \{-i, 0, i\}$$

$\vdots$

What is $\bigcup_{i=1}^{\infty} A_i$ and $\bigcap_{i=1}^{\infty} A_i$.?

$\bigcup_{i=1}^{\infty} A_i = \{x : x$ belongs to at least one of $A_i\}$.

Any integer belongs to some A_i.

Take an integer n. Then $n \in A_n = \{-n, 0, n\}$.

So every integer belongs to $\bigcup_{i=1}^{\infty} A_i$

so $\mathbb{Z} = \bigcup_{i=1}^{\infty} A_i$

$\bigcap_{i=1}^{\infty} A_i = \{x : x$ belongs to every $A_i\}$. Only zero has

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this property. $0 \in A_i$ for all i, $\bigcap_{i=1}^{\infty} A_i = \{0\}$.

Instead of numbering the sets, one can label them with elements of any set I called an index set.

$\bigcup_{i \in I} A_i$ is the set of elements that belong to *at least one* of the sets A_i.

$\bigcap_{i \in I} A_i$ is the set of elements that belong to *every one* of the sets A_i.

$A_1, A_2, \ldots$

what about A_r for every real number r.

$\bigcup_{i=1}^{\infty} A_i \xleftarrow{\text{Instead}} \bigcup_{i \in \mathbb{N}} A_i$

$\cdots \cup A_{-3} \cup A_{-2} \cup A_{-1} \cup A_0 \cup \cdots$

$\bigcup_{i \in \mathbb{Z}} A_i$

Index sets example

Let C be the set of Counties in the state of Connecticut (there are 8 of these). For each county $c \in C$, let $T(c)$ be the set of Towns in that County.

For example, if c is Tolland County, then the elements of $T(c)$ are Andover, Bolton, Columbia, Coventry, Ellington, Hebron, Mansfield, Somers, Stafford, Tolland, Union, Vernon, and Willington.

What is $\bigcup_{c \in C} T(c)$?

$T_c = T(c) = \{\text{towns in county } c\}$.

$\bigcup_{c \in C} T(c) = \{\text{elements in any of the counties}\}$

$= \{\text{all towns in CT}\}$

$\bigcap_{c \in C} T(c) = \emptyset$

Index sets

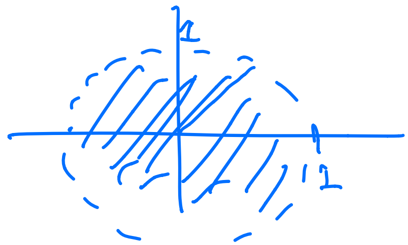
Let

$$\mathbb{R}_+ = \{r : r \in \mathbb{R}, r > 0\}. \quad = (0, \infty)$$

For every real number $r \in \mathbb{R}_+$, let

$$A_r = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < r^2\}.$$

A_1:



A_r = interior of circle of radius r in plane

A_2:

$x^2 + y^2 < 4$

Index sets

What is $\bigcap_{r \in \mathbb{R}_+} A_r$?

What, if any, pair (x,y) is in A_r for every $r>0$?

$x^2+y^2<r^2$ for every $r \in \mathbb{R}^+$

$r>0$

Inside circle of radius r no matter how small r gets.

$(0,0) \in \bigcap_{r \in \mathbb{R}_+} A_r$

$(0,0) \in A_r$ because

$0^2+0^2<r^2$

$0<r^2$

(x,y) choose $r' < x^2+y^2$

then $(x,y) \notin A_{r'}$, so not in $\bigcap_{r \in \mathbb{R}^+} A_r$.

$$\bigcap_{r \in \mathbb{R}_+} A_r = \{(0,0)\}$$

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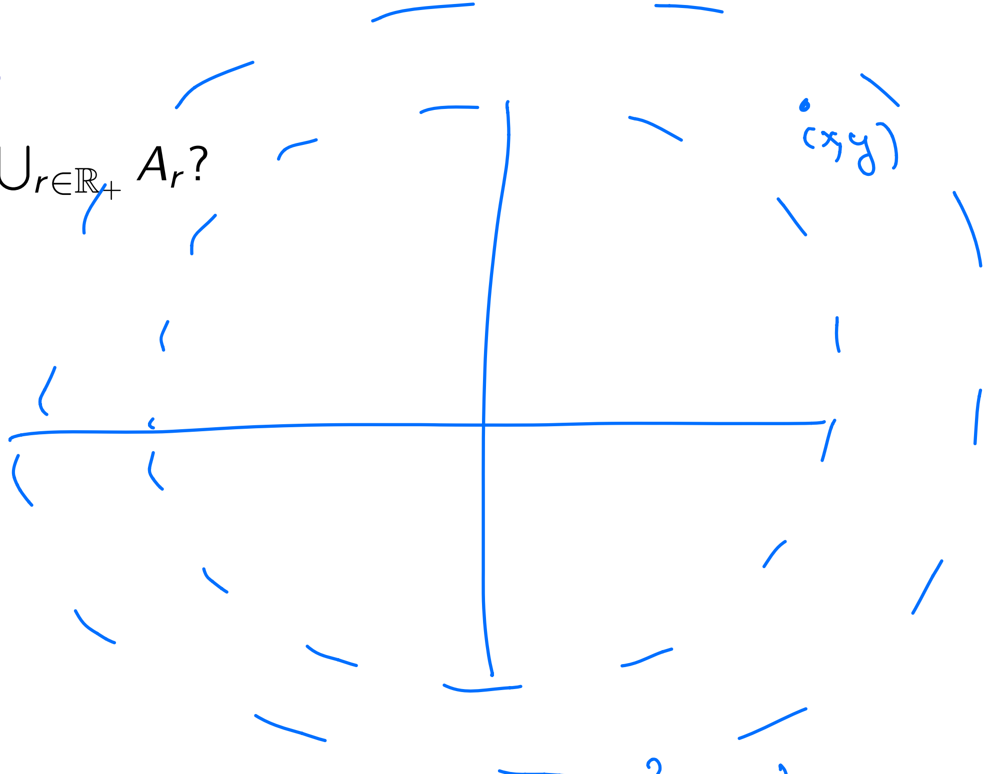
What is $\bigcup_{r \in \mathbb{R}_+} A_r$?



Every point $(x,y) \in A_r$ once $r^2 > x^2 + y^2$

So every point in $\mathbb{R}^2$ in $\bigcup_{r \in \mathbb{R}_+} A_r$

So $\mathbb{R}^2 = \bigcup_{r \in \mathbb{R}_+} A_r$

Example

What is $\bigcap_{i\in\mathbb{N}}[0, i+1]$?

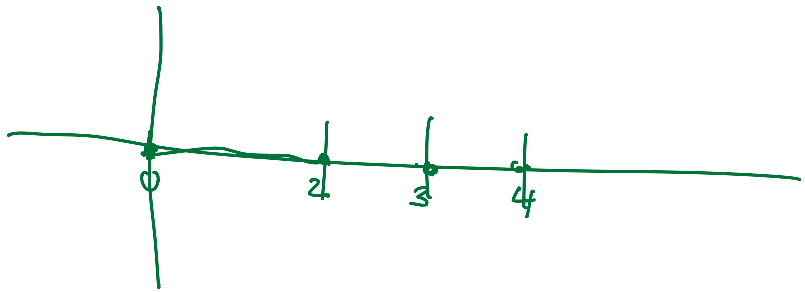
$A_1 = [0,2]$

$A_2 = [0,3]$

$A_3 = [0,4]$

$\vdots$

$[0,2] \subseteq \mathbb{R}$



$A_1 \subseteq \bigcap_{i\in\mathbb{N}} [0,i+1]$

$=$

$A_1 \subseteq A_2 \subseteq A_3 \cdots$

$x \in A_1$ is in $\bigcap_{i\in\mathbb{N}} [0,i+1]$

if $x \in \bigcap_{i\in\mathbb{N}} [0,i+1]$ then $x \in [0,2]$

$A_1 = \bigcap_{i\in\mathbb{N}} [0,i+1]$

Example

Suppose that I and J are sets, that $J \neq \emptyset$, and that $J \subseteq I$. Is

$$\bigcap_{a \in I} A_a \subseteq \bigcap_{a \in J} A_a?$$

We know nothing about the sets A_a

Explain.

$I = \{1, 2, 3\}$

$J = \{1, 2\}$

$J \neq \emptyset$ and $J \subseteq I$.

We have sets A_1, A_2, A_3

$$\bigcap_{a \in I} A_a = A_1 \cap A_2 \cap A_3$$

$$\bigcap_{a \in J} A_a = A_1 \cap A_2$$

In this case the question is:

Is $A_1 \cap A_2 \cap A_3 \subseteq A_1 \cap A_2$?

Left side $= \{x : x \in A_1 \text{ and } x \in A_2 \text{ and } x \in A_3\}$

Right side $= \{x : x \in A_1 \text{ and } x \in A_2\}$

so therefore $A_1 \cap A_2 \cap A_3 \subseteq A_1 \cap A_2$

$I = \mathbb{Z}$ $\qquad J = \{x : x \in \mathbb{Z}, x \text{ is even}\}$

$J \subseteq I$

Is $\underline{\bigcap_{a \in \mathbb{Z}}} A_a \subseteq \underline{\bigcap_{a \in J}} A_a$?

LHS is $\ldots A_{-3} \cap A_{-2} \cap A_{-1} \cap A_0 \cap A_1 \cap \ldots$

$= \{x : x \in A_n \text{ for every } n \in \mathbb{Z}\}$

RHS is $\ldots A_{-4} \cap A_{-2} \cap A_0 \cap A_2 \cap A_4 \cap \ldots$

$= \{x : x \in A_n \text{ for every even } n \in \mathbb{Z}\}$

Belonging to every A_n for $n \in \mathbb{Z}$ forces you to belong to every A_n for $n \in \mathbb{Z}$ and n even.

General case:

if x belongs to A_a for every $a \in I$, then, since $J \subseteq I$, x belongs to A_a for every $a \in J$. therefore

$$\bigcap_{a \in I} A_a \subseteq \bigcap_{a \in J} A_a$$