The Power Set of a Set

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Definition

Definition: If A is a set, the **power set** of A, written $\mathcal{P}(A)$, is the set whose elements are all subsets of A. In set builder notation,

$$\mathcal{P}(A) = \{X : X \subseteq A\}$$

Example

$$A = \{0, 1, 3\}$$

# of subsets of $A = 2^3 = 8$ $\quad |A| = 3$

$$\mathcal{P}(A) = \{\emptyset, \{0\}, \{1\}, \{3\}, \{0, 1\}, \{0, 3\}, \{1, 3\}, \{0, 1, 3\}\}$$

1 $\emptyset$

$\{0\}$

3 $\{1\}$

$\{3\}$

$\{0,1\}$

3 $\{0,3\}$

$\{1,3\}$

$\{0,1,3\}$

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Example

$$\mathcal{P}(\emptyset) = \{\emptyset\}$$

Notice that $|\emptyset| = 0$ and $|\mathcal{P}(\emptyset)| = 2^0 = 1$.

$\mathcal{P}(\phi) = \{\phi\}$

$|\phi| = 0$

$|\mathcal{P}(\phi)| =$ # of subsets of $\phi = 2^{|\phi|} =$

$2^0 = \underline{1}$

$\mathcal{P}(\{\phi\}) = \{\phi, \{\phi\}\}$

$|\mathcal{P}(\{\phi\})| = |\mathcal{P}(\mathcal{P}(\phi))|$

$= 2^1 = 2$

Example

$$\mathcal{P}(\{a\}) = \{\emptyset, \{a\}\}$$

$|\{a\}| = 1$

$|\mathcal{P}(\{a\})| = 2 = 2^1$

Example – some common mistakes

$\mathcal{P}(1)$ makes no sense because 1 is not a set.

$\mathcal{P}(\{1\})$ allowed

$\mathcal{P}(2,3,4)$ – meaningless

$\mathcal{P}(\{2,3,4\})$ – means something

Example – some common mistakes 2

$\mathcal{P}(\{1,\{1,2\}\}) = \{\emptyset, \{1\}, \{\{1,2\}\}, \{1,\{1,2\}\}\}$. Notice that $\underline{\{1,2\}}$ is not an element of $\mathcal{P}(\{1,\{1,2\}\})$ but $\{\{1,2\}\}$ is.

$\{1, \{1,2\}\}$

$\wp(\{1,\{1,2\}\}) = \{ \quad \}$

$\{1\} \subseteq \{1, \{1,2\}\}$ $\qquad$ $\{\{1,2\}\} \subseteq \{1, \{1,2\}\}$

$\phi \subseteq \{1, \{1,2\}\}$

$\{1, \{1,2\}\} \subseteq \{1, \{1,2\}\}$

Infinite case

THe power set $\mathcal{P}(\mathbb{N})$ is very large and can be identified with infinite sequences of I's and O's.

$$\wp(\mathbb{N}) \ni \{1, 3, 11, 14\}$$

$$\{1, 3, 11, 14\} \in \wp(\mathbb{N})$$

$$\{2x : x \in \mathbb{N}\} \in \wp(\mathbb{N})$$

$$\{1, 4, 9, 16, 32, \ldots\} \in \wp(\mathbb{N})$$

$$\mathbb{N} = \{1, 2, 3, 4, \ldots\}$$

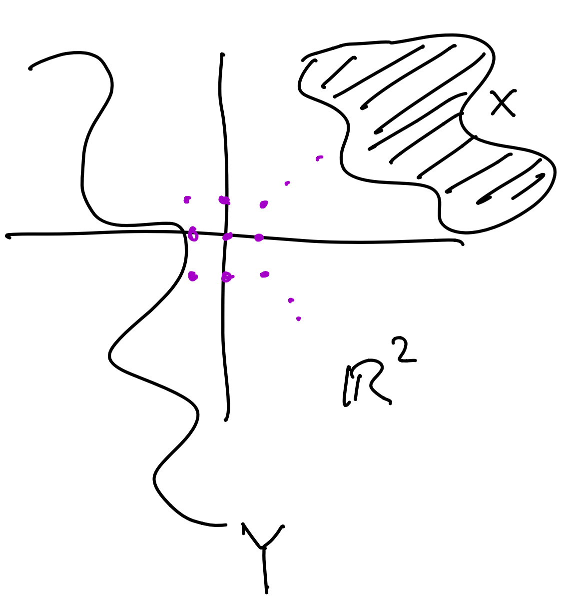
$$(I, I, O, I, \underbrace{O, O, \ldots}_{\text{all } O\text{'s}}) \longleftrightarrow \{1, 2, 4\}$$

$$110111001000\ldots \longleftrightarrow \text{a subset of } \mathbb{N}$$

1 2 4 5 6 9

The set $\mathcal{P}(\mathbb{R}^2)$

$\mathcal{P}(\mathbb{R}^2)$ is huge and includes every graph of every function plus lots of other things, more than we can really comprehend.



$X \in \wp(\mathbb{R}^2)$

$Y \in \wp(\mathbb{R}^2)$

$$\mathbb{Z} \times \mathbb{Z} = \left\{ (x, y) : \begin{array}{l} x \in \mathbb{Z} \\ y \in \mathbb{Z} \end{array} \right\}$$

$\mathbb{Z} \times \mathbb{Z} \in \wp(\mathbb{R}^2)$

Problem 1.4.15

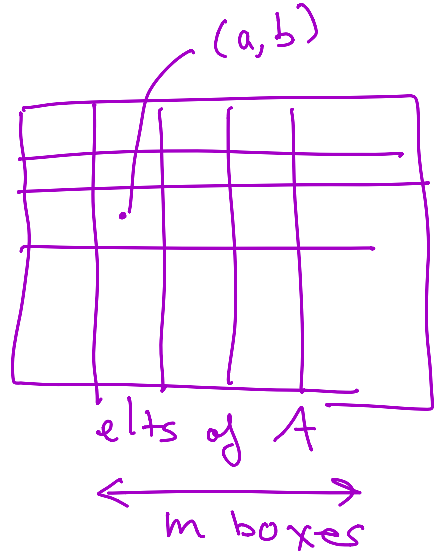
What is $|\mathcal{P}(A \times B)|$ if $|A| = m$ and $|B| = n$?

$$A \times B = \{(a,b) : a \in A, b \in B\}$$

$$|A \times B| = mn = |A||B|$$

$$\mathcal{P}(A \times B) = \{X : X \subseteq A \times B\}$$

$$|\mathcal{P}(A \times B)| = 2^{|A \times B|} = 2^{mn}$$



total of mn boxes

e.g. $m = n = 2$

$|A| = \{a, b\}$ $\quad B = \{0, 1\}$.

$$A \times B = \{(a,0), (a,1), (b,0), (b,1)\}$$

$$|\mathcal{P}(A \times B)| = 2^4 = 16$$

samples: $\{(a,0), (b,0), (b,1)\}$
$\qquad \in \mathcal{P}(A \times B)$

$$\{(b,0),(b,1)\} \subseteq A \times B$$

and so $\{(b,0),(b,1)\} \in \wp(A \times B)$.