

# Counting Subsets

# Counting Subsets

# Counting subsets of a finite set

**Theorem:** A finite set with  $n$  elements has  $2^n$  subsets.

$\{1, 2\}$  has 4 subsets:

- ▶ the empty set  $\emptyset$ ,
- ▶ the one element sets  $\{1\}$  and  $\{2\}$ ,
- ▶ the two element set  $\{1, 2\}$ .

$\emptyset$  is a subset of every set

$$\begin{aligned}\{1\} &\subseteq \{1, 2\} \\ \{2\} &\subseteq \{1, 2\}\end{aligned}$$

$$\{1, 2\} \subseteq \{1, 2\}$$

$$2^2 = 4$$

# Counting subsets

The book gives one explanation for why this is true on page 13. We will give a slightly different one.

$\{a, b, c\}$

$\emptyset$   
 $\{a\}, \{b\}, \{c\}$   
 $\{a, b\}, \{a, c\}, \{b, c\}$   
 $\{a, b, c\}$

$$\begin{array}{r} 1 \\ + 3 \\ + 3 \\ + 1 \\ \hline 8 \end{array}$$

$$2^3 = 8.$$

# Counting subsets

Suppose we have a finite set  $A$  with  $n$  elements. We will list the elements as  $a_1, a_2, \dots, a_n$ .

$$A = \{\underline{a}_1, \underline{a}_2, \dots, \underline{a}_n\}.$$

$$A = \{1, 2, 3\}$$

$$a_1 = 1$$

$$a_2 = 2$$

$$a_3 = 3$$

Here, we've decided to put the elements of  $A$  in order, but it doesn't matter what order you use.

$$A = \left\{ \begin{array}{l} \text{hippo,} \\ \text{elephant,} \\ \text{giraffe,} \\ \text{rhino} \end{array} \right\}$$

$$\begin{array}{l} a_1 = \text{hippo} \\ a_2 = \text{elephant} \\ a_3 = \text{giraffe} \\ a_4 = \text{rhino} \end{array}$$

# Counting subsets

A subset  $B$  of  $A$  is determined by going through the elements of  $A$  and marking each element as either “in” or “out” of the subset. So we can describe a subset of  $A$  by giving a list

$I, I, O, I, O, \dots, I$

where we have an  $I$  if that element is in the subset, or an  $O$  if it isn't.

$\{ \text{hippo}, \text{elephant}, \text{giraffe}, \text{rhino} \}$   
I I O O



$\{ \text{hippo}, \text{elephant} \}$

## Counting example

Suppose  $A = \{-1, 4, 7, 8\}$ . We put the elements of  $A$  in that order, so  $a_1 = -1$ ,  $a_2 = 4$ ,  $a_3 = 7$ , and  $a_4 = 8$ . Let  $B = \{-1, 7\}$  so that  $B \subseteq A$ .

Then  $B$  corresponds to the list

$$\begin{array}{cccc} -1 & 4 & 7 & 8 \\ I, & O, & I, & O \end{array} \iff \{-1, 7\}.$$

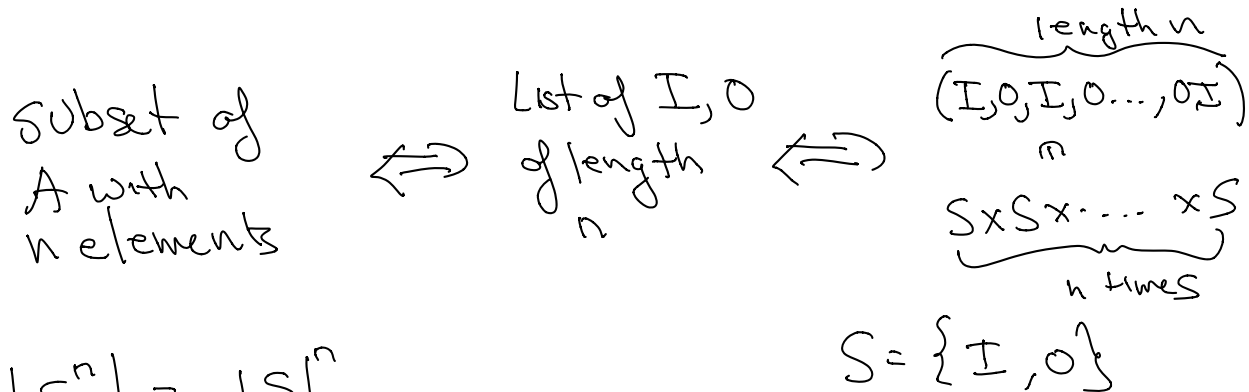
since  $-1$  is IN  $B$ ,  $4$  is OUT of  $B$ ,  $7$  is IN  $B$ , and  $8$  is OUT of  $B$ .

The list  $O, I, O, O$  corresponds to the subset  $\{4\}$  since only  $4$  is IN this set.

# Counting subsets

**Theorem:** The number of subsets of a set  $A$  with  $n$  elements is the same as the number of ordered sequences of  $I$  and  $O$  of length  $n$ , and this number is  $2^n$ .

**Proof:** Let  $S = \{I, O\}$ . We've seen above how a sequence of  $I$  and  $O$  correspond to a subset. The set of sequences of  $I$  and  $O$  of length  $n$  is exactly  $S^n$ . By our earlier counting result,  $|S^n| = |S|^n = 2^n$ .





# Subsets of a set with  $n$  elements is  $2^n$ .