Counting Subsets

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Counting subsets of a finite set

Theorem: A finite set with n elements has 2^n subsets.

$\{1,2\}$ has 4 subsets:

- the empty set $\emptyset$,
- the one element sets $\{1\}$ and $\{2\}$,
- the two element set $\{1,2\}$.

ϕ is a subset of every set

$\{1\} \subseteq \{1,2\}$

$\{2\} \subseteq \{1,2\}$

$\{1,2\} \subseteq \{1,2\}$

$2^2 = 4$

Counting subsets

The book gives one explanation for why this is true on page 13. We will give a slighly different one.

$\{a, b, c\}$

$\emptyset$

$\{a\}, \{b\}, \{c\}$

$\{a, b\}, \{a, c\}, \{b, c\}$

$\{a, b, c\}$

$2^3 = 8.$

$$\begin{array}{r} 1 \\ + \cancel{2}3 \\ + 3 \\ + 1 \\ \hline 8 \end{array}$$

Counting subsets

Suppose we have a finite set A with n elements. We will list the elements as $a_1, a_2, \dots, a_n$.

$A = \{1, 2, 3\}$

$a_1 = 1$

$a_2 = 2$

$a_3 = 3$

$$A = \{a_1, a_2, \dots, a_n\}.$$

Here, we've decided to put the elements of A in order, but it doesn't matter what order you use.

$A = \{$hippo, elephant, giraffe, rhino$\}$

$a_1 =$ hippo

$a_2 =$ elephant

$a_3 =$ giraffe

$a_4 =$ rhino

Counting subsets

A subset B of A is determined by going through the elements of A and marking each element as either "in" our "out" of the subset. So we can describe a subset of A by giving a list

$$I, I, O, I, O, \ldots, I$$

where we have an I if that element is in the subset, or an O if it isn't.

$$\{\text{hippo}, \text{elephant}, \text{giraffe}, \text{rhino}\}$$

$$\begin{array}{cccc} I & I & O & O \end{array}$$

$$\updownarrow$$

$$\{\text{hippo}, \text{elephant}\}$$

Counting example

Suppose $A = \{-1, 4, 7, 8\}$. We put the elements of A in that order, so $a_1 = -1, a_2 = 4, a_3 = 7$, and $a_4 = 8$. Let $B = \{-1, 7\}$ so that $B \subseteq A$.

Then B corresponds to the list

$$\begin{array}{cccc} -1 & 4 & 7 & 8 \\ I, & O, & I, & O \end{array} \iff \{-1, 7\}.$$

since -1 is IN B, 4 is OUT of B, 7 is IN B, and 8 is OUT of B.

The list O, I, O, O corresponds to the subset $\{4\}$ since only 4 is IN this set.

Counting subsets

Theorem: The number of subsets of a set A with n elements is the same as the number of ordered sequences of I and O of length n, and this number is 2^n.

Proof: Let $S = \{I, O\}$. We've seen above how a sequence of I and O correspond to a subset. The set of sequences of I and O of length n is exactly S^n. By our earlier counting result, $|S^n| = |S|^n = 2^n$.

Subset of A with n elements $\iff$ List of I, O of length n $\iff$ $\overbrace{(I, O, I, O \ldots, O, I)}^{\text{length } n}$

$\cap$

$\underbrace{S \times S \times \cdots \times S}_{n \text{ times}}$

$S = \{I, O\}$

$|S^n| = |S|^n$

$|S^n| = 2^n$

subsets of a set with n elements is 2^n.