Cartesian Products

Cartesian Products

Definition

Definition: The Cartesian Product $A \times B$ of two sets A and B is the set of ordered pairs (a, b) where $a \in A$ and $b \in B$.

$$A \times B = \{(a, b) : a \in A, b \in B\}$$

Example

$A = \{1, 2, 3\}$ and $B = \{x, y\}$. What is $A \times B$?

$$A \times B = \{(a, b) : a \in A, b \in B\}.$$

$(1, x) \in A \times B$

$(2, x) \in A \times B$

$(3, x) \in A \times B$

$(1, y) \in A \times B$

$(2, y) \in A \times B$

$(3, y) \in A \times B$

$$A \times B = \{(1, x), (2, x), (3, x), (1, y), (2, y), (3, y)\}$$

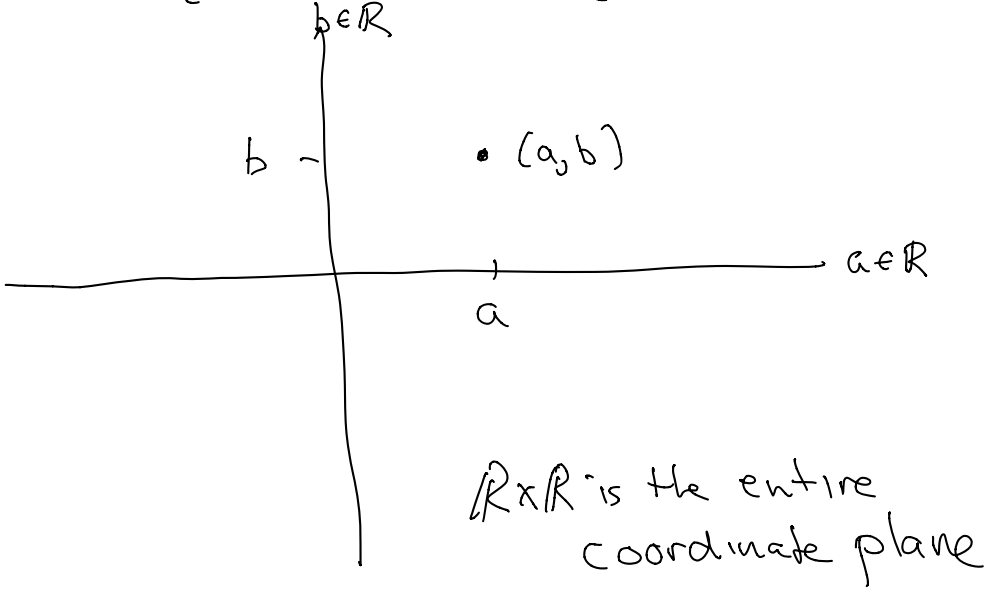
$$|A \times B| = 6$$

Example

$$\mathbb{R} \times \mathbb{R} = \{(a, b) : a \in \mathbb{R}, b \in \mathbb{R}\}$$

$\mathbb{R} = \{x : x \text{ is a real number}\}$

$\mathbb{R} \times \mathbb{R} = \{(a, b) \mid a, b \in \mathbb{R}\}$



$\mathbb{R} \times \mathbb{R}$ is the entire coordinate plane

Example

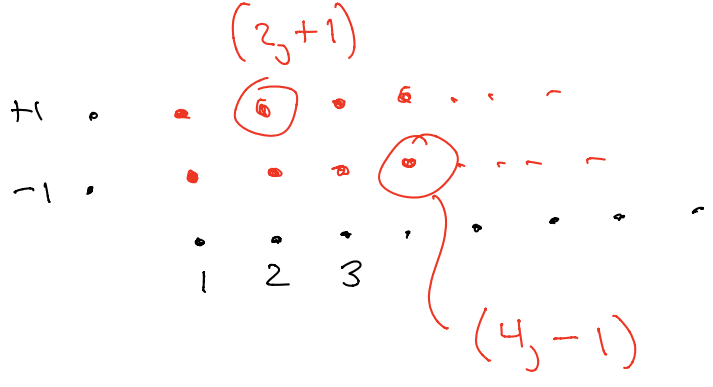
What is $\mathbb{N} \times \{-1, 1\}$?

$\mathbb{N} = \{1, 2, 3, \ldots\}$

$\mathbb{N} \times \{-1, 1\} = \{(a, b) \mid a \in \mathbb{N},\ b \in \{+1, -1\}\}$

$\mathbb{N} \times \{-1, 1\} = \{(1, +1), (2, +1), (3, +1), \ldots\}$

$\{(1, -1), (2, -1), (3, -1), \ldots\}$



Example

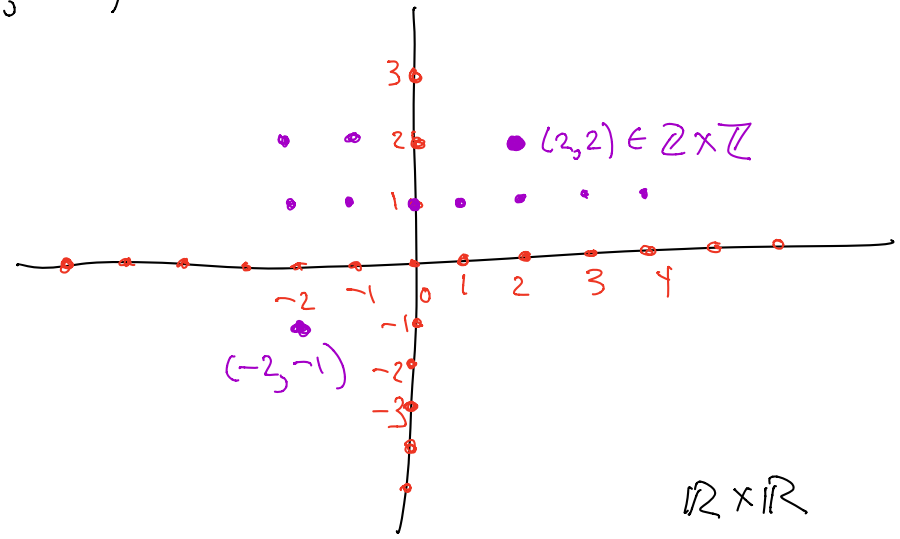
$\mathbb{Z} \times \mathbb{Z}$

$\mathbb{Z} = \{\ldots, -5, -4, -3, -2, \ldots\ldots\ldots 5, 6, 7, \ldots\}$

$\mathbb{Z} \times \mathbb{Z} = \{(a, b) : a, b \in \mathbb{Z}\}$

$(3, -2) \in \mathbb{Z} \times \mathbb{Z}$

$(-14, 0) \in \mathbb{Z} \times \mathbb{Z}$



$\mathbb{R} \times \mathbb{R}$

Example

$\mathbb{N} \times (\mathbb{N} \times \mathbb{N})$ vs $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$

$\mathbb{N} \times (\mathbb{N} \times \mathbb{N}) = \{(a, b) \mid a \in \mathbb{N},\ b \in \mathbb{N} \times \mathbb{N}\}$

$A = \mathbb{N}$, $B = \mathbb{N} \times \mathbb{N}$

$(1, (2, 3)) \in \mathbb{N} \times (\mathbb{N} \times \mathbb{N})$

$\mathbb{N} \times (\mathbb{N} \times \mathbb{N}) = \{(a, (b, c)) : a, b, c \in \mathbb{N}\}$

$\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ cartesian "power"

$= \{(a, b, c) : a, b, c \in \mathbb{N}\}$

$\mathbb{N}^3 = \mathbb{N} \times \mathbb{N} \times \mathbb{N}$

$= \{\text{ordered triples}\}$

$(1, 13, 2) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$

$A = \mathbb{N}$ $B = \{1, 2\}$ $C = \mathbb{R}$

$A \times B \times C = \{(a, b, c) : a \in \mathbb{N},\ b \in \{1, 2\},\ c \in \mathbb{R}\}$

$(4, 2, \pi) \in A \times B \times C$

Cartesian Powers

$A = \{H, T\}$. What is A^4? What is $|A^4|$?.

$A = \{H, T\}$

$A^4 = A \times A \times A \times A = \{(a, b, c, d) : a, b, c, d \in A\}$.

$(H, T, T, H) \in A^4$

$(T, T, T, T) \in A^4$

$(H, H, H, H) \in A^4$

A^4 has same # of elements as $A^2 \times A^2$

$((H, T), (T, H)) \leftrightarrow (H, T, T, H)$

$|A^4| = |A|^4$

$|A^2| = |A \times A| = |A|^2$

$|A^2| = |A|^2$

$|A^4| = |A|^4$

$|A^n| = |A|^n$.

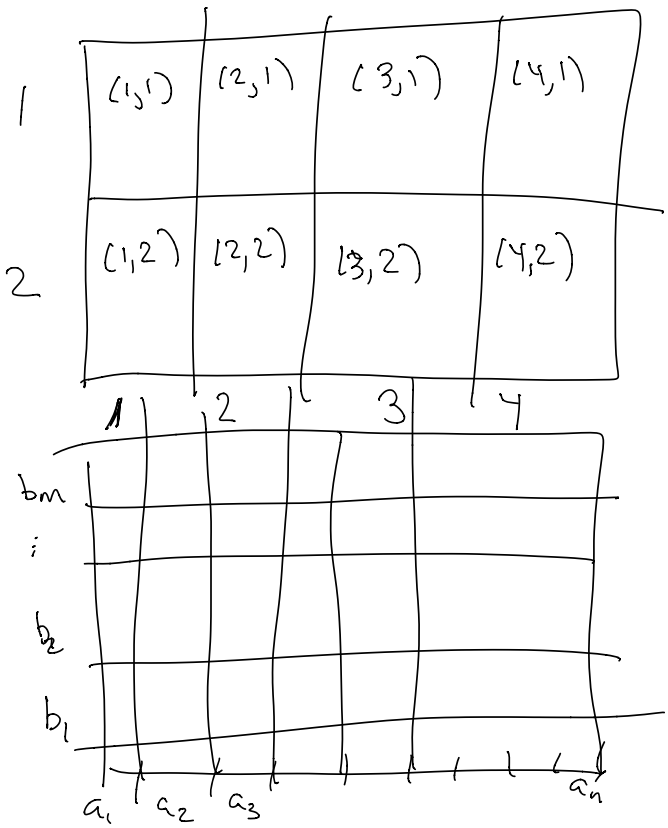
$|A \times B \times C| = |A||B||C|$.

How many elements are in $A \times B$ if A and B are finite?

$A \times B = \{(a,b) : a \in A \quad b \in B\}$.

$A = \{1,2,3,4\}$

$B = \{1,2\}$



$|A \times B| = 8$

$= |A||B|$

rectangles $(A \times B)$

$= |A||B|$

Theorem: if A and B are finite then $|A \times B| = |A||B|$.