1. Modules

Modules: Basics

How to think of modules

- Modules are to rings as vector spaces are to fields.
- Modules are to rings as sets with group actions are to groups.

Definition of (left) modules

Definition: Let R be a ring (for now, not necessarily commutative and not necessarily having a unit). A *left R-module* is an abelian group M together with a map $R \times M \to M$ (written $(r, m) \mapsto rm$) such that:

- $r(m_1 + m_2) = rm_1 + rm_2$
- $(r_1 + r_2)m = r_1 m + r_2 m$
- $r_1(r_2 m) = (r_1 r_2)m$

If R has a unit element 1, we also require $1m = m$ for all $m \in M$.

Right modules

A right module is defined by a map $M \times R \to M$ and written $(m, r) \mapsto mr$ and satisfying the property

$$(mr_1)r_2 = m(r_1 r_2).$$

If R is not commutative, these really are different, since for a left module:

- $r_1 r_2$ acts by "first r_2, then r_1

while for a right module

- $r_1 r_2$ acts by "first r_1, then r_2."

Left and Right modules

If R is commutative, and M is a left R-module, then we can define a right R module M' with the same underlying abelian group M and by defining $m'r = (rm)'$. This works because

$$(m'r_1)r_2 = (r_1m)'r_2 = (r_2(r_1m))' = ((r_2r_1)m)' = ((r_1r_2)m)' = m'(r_1r_2)$$

Remarks

Vector spaces

If R is a field, then a left (or right) R-module is the same as a vector space.

Another definition

If M is an abelian group, and R is a ring, then a left R-module structure on M is the same as a ring map

$$R \to \mathrm{End}(M).$$

If ϕ_r is the endomorphism associated to $r \in R$, then $rm = \phi_r(m)$. The associativity comes from defining the ring structure on

$$\mathrm{End}(M)$$

as the usual composition of functions:

$$\phi_{r_1 r_2} = \phi_{r_1} \circ \phi_{r_2}.$$

Submodules

Definition: If M is a left R-module, then a submodule N of M is a subgroup with the property that, if $n \in N$, then $rn \in N$ for all $r \in R$.

Observation: A ring R is a left module over itself by ring multiplication. The (left) ideals of R are *exactly the left submodules of* R.

Essential examples

Rings as modules over themselves

- Every ring R is a left module over itself. The submodules of R are the left ideals.
- R is also a right module over itself, with the right ideals being the right submodules.

If F is a field and $n > 1$, let $R = M_n(F)$ be the $n \times n$ matrix ring over F. The matrices with arbitrary first column and zeros elsewhere form a left ideal J and therefore a left submodule of R as left R-module. But J is *not* a right R-submodule.

A field F is a one-dimensional vector space over itself, and a commutative ring R is a module (left and right) over itself with the ideals of R being the submodules.

Free modules

Let R be a ring with unity and let $n \geq 1$ be a positive integer. Then

$$R^n = \{(r_1, \ldots, r_n) : r_i \in R \text{ for } i = 1, \ldots, n\}$$

is an R module with componentwise addition and multiplication given by $r(r_1, \ldots, r_n) = (rr_1, \ldots, rr_n)$.

This is called the *free R-module of rank n*.

Free modules and vector spaces

- If R is a field, the free R-module of rank n is an n-dimensional vector space.
- The submodules of a finite dimensional vector space are all subspaces which are copies of R^k for $k \leq n$.
- For more general R the picture is more complicated. Let $R = \mathbb{Z}$ and $M = \mathbb{Z}^2$. Then:
  - $\{(n, 0) : n \in \mathbb{Z}\}$ is a submodule of M which "looks like" a subspace.
  - $2M = \{(a, b) : a, b \in 2\mathbb{Z}\}$ is a submodule of M which does not.

Change of rings (restriction of scalars)

- An abelian group M may be an R module for different rings R. For example:
  - $\mathbb{Q}$ is a module over $\mathbb{Q}$, where it is a one dimensional vector space and its only $\mathbb{Q}$-submodules are 0 and itself.
  - $\mathbb{Q}$ is a module over $\mathbb{Z}$, and it has many $\mathbb{Z}$-submodules, such as $\mathbb{Z}[1/2]$.

More generally, if $R \subset S$ is a subring, and M is an S-module, then it is an R-module. This is called *restriction of scalars*.

$\mathbb{Z}$-modules are the same as abelian groups

Let M be an abelian group. Then it is automatically a $\mathbb{Z}$-module where we define

$$nx = \overbrace{x + x + \cdots + x}^{n}.$$

Furthermore, given any $\mathbb{Z}$-module, it must be the case that

$$nx = (\overbrace{1 + 1 + \cdots + 1}^{n})x = \overbrace{x + x + \cdots + x}^{n}.$$

(Note: this is why we require $1x = x$ when R is a ring with unity in the module axioms).

Further, submodules of M (as $\mathbb{Z}$-module) are just the subgroups of M (as abelian group).

Change of rings (quotients)

Suppose that M is a left R module and $I \subset R$ is a two-sided ideal with the property that, for all $y \in I$, and all $x \in M$, we have $yx = 0$. In this case we say that I annihilates M or that $IM = 0$.

With this hypothesis, we may view M as an R/I module by defining $(r + I)m = rm$ for any coset representative $r + I \in R/I$. This is well-defined since two different coset representatives r, r' satisfy $r' = r + i$ for some $i \in I$ and therefore $r'm = (r + i)m = rm$ since $im = 0$.

If M is an abelian group and $m \in Z$ is a positive integer such that $mM = 0$, then M can be viewed as a module over $\mathbb{Z}/m\mathbb{Z}$ by this process.

This operation is a special case of a general operation called *base change* or *extension of scalars* that we will study in more detail later.

Modules over $F[x]$

Basic construction

Let F be a field, let V be a vector space over F, and let $T : V \to V$ be an F-linear transformation. Define a homomorphism

$$F[x] \to \mathrm{End}(V)$$

by sending

$$x^k \mapsto T^k = \overbrace{T \circ T \circ \cdots \circ T}^{n}.$$

This construction makes V into a module for $F[x]$ *which depends on the choice of the linear transformation* T.

Polynomials and linear transformations

Let $V = F^2$ and let T be the linear transformation given by the matrix

$$T = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

If e_0 and e_1 are the standard basis elements of F^2 then

$$\begin{aligned} Te_0 &= & e_1 \\ T^2 e_0 = Te_1 = e_0 + e_1 = e_0 + Te_0 &= & (1 + T)e_0 \end{aligned}$$

Polynomials and linear transformations continued

Therefore $(T^2 - T - 1)e_0 = 0$ and

$$(T^2 - T - 1)e_1 = (T^2 - T - 1)Te_0 = T(T^2 - T - 1)e_0 = 0$$

so the polynomial $x^2 - x - 1$ is in the kernel of the map from $F[x] \to \mathrm{End}(V)$.

By the base change construction above this means that V can be viewed as a module over $F[x]/(x^2 - x - 1)$.

Characterization of $F[x]$ modules

We saw above that, given an F-vector space V with a linear transformation T, we get an $F[x]$ module where x acts on V through T.

Conversely, suppose that M is an module over $F[x]$. Then M is an F vector space (via the restriction of scalars from $F[x]$ to F). Furthermore, the element $x \in F[x]$ acts on M as an F-linear transformation because that's what the module axioms amount to.

Therefore there is an equivalence between

$$\{F[x] - \text{modules}\}$$

$$\Updownarrow$$

$$\{\text{vector spaces } V \text{ over } F \text{ with a given linear map } T : V \to V\}$$

Submodules of $F[x]$ modules

In the correspondence above, a submodule of an $F[x]$ module M corresponds to a subspace $W \subset V$ that is *preserved by* T, meaning $TW \subset W$.

Thus, not all subspaces of V correspond to submodules.

In the example given earlier, the only T-stable proper subspace of V is the zero subspace.

If we consider instead the linear map on F^2 satisfying $Ue_0 = 0$ and $Ue_1 = e_0$, then the one dimensional subspace spanned by e_0 is U-stable and F^2 viewed as an $F[x]$ module via U has a submodule corresponding to that subspace.

Checking the submodule property

Proposition: A subset N of a left R-module M is a submodule if it is nonempty and, for all $x, y \in N$ and $r \in R$, we have $x + ry \in N$. Alternatively, if N is a subgroup of the abelian group M and $rN \subset N$ for all $r \in R$ then N is a submodule.

Algebras

Definition: Let R be a commutative ring with unity. An R-algebra is a (not necessarily commutative) ring S with a ring homomorphism $f : R \to S$ carrying 1_R to 1_S such that $f(R)$ is in the center of S.

The polynomial ring $F[x]$ is an F-algebra, as is the matrix ring $M_n(F)$ where the homomorphism $f : F \to M_n(F)$ embeds F as the diagonal matrices. More generally, any F-algebra A, where F is a field, contains F in its center and the identites of A and F are the same.

The ring $\mathbb{Z}/p\mathbb{Z}$ is a $\mathbb{Z}$-algebra. In fact any ring S with 1 is a $\mathbb{Z}$ algebra by the map sending $n \in \mathbb{Z}$ to $n1_S$.

The ring $\mathbb{Q}[x]$ is a $\mathbb{Z}[x]$ algebra.

We typically omit the explicit map f and just think of R as "contained in" A; this can be misleading since f doesn't need to be injective, but it works in practice.

Algebra morphisms

Definition: A map of R-algebras $f : A \to B$ is a ring homomorphism that is R-linear in the sense that $f(ra) = rf(a)$ for all $r \in R$ and $a \in A$.

Any homomorphism of rings with unity is a $\mathbb{Z}$-algebra morphism.

Modules Homomorphisms, Quotient Modules, and Mapping Properties

Module homomorphisms

Definition: Let R be a ring and let M and N be (left) R-modules. A function $f : M \to N$ is an R-module homomorphism if:

- it is a homomorphism between the abelian group structures on M and N
- it is R-linear, meaning $f(rm) = rf(m)$ for all $r \in R$.

Note that, if R is a field, then M and N are vector spaces and an R-module homomorphism is just a linear map.

A module isomorphism is a bijective homomorphism.

We let $\mathrm{Hom}_R(M, N)$ denote the set of R-module homomorphisms from M to N.

Kernels and images

Let R be a ring and let M and N be R-modules. Let $f : M \to N$ be a homomorphism.

- Let $\ker(f) = \{m \in M : f(m) = 0\}$ (the *kernel* of f). This is a submodule of M.
- Let $f(M) \subset N$ be the image of f. Then $f(M)$ is a submodule of N.

Quotient modules

Let M be an R module and let $N \subset M$ be a submodule.

Definition: Let M/N be the quotient abelian group. Then M/N is an R-module where R acts on cosets by

$$r(x+N) = rx + N.$$

This is called the quotient module of M by N.

The R-module structure is well defined because if $x + N = y + N$, then $x = y + n$ for some $n \in N$, and $rx = ry + rn$. Since N is a submodule, $rn \in N$ so $rx + N = ry + N$.

Notice that N can be any submodule, there is no "normality" condition like for groups.

There is always a "projection" homomorphism $\pi : M \to M/N$ defined by $\pi(m) = m + N$ which has kernel N.

Sums of modules

If A and B are submodules of a module M, then $A+B$ is the smallest submodule of M containing both A and B. Alternatively it is:

$$A+B=\{a+b : a\in A, b\in B\}$$

Mapping Properties

Let M, N, and K be R modules, and let $f : M \to K$ be a homomorphism with $N \subset \ker(f)$. Then there is a unique homomorphism $\overline{f} : M/N \to K$ making this diagram commutative:

$$\begin{tikzcd} M \arrow[d, "\pi"] \arrow[dr, "f"] & \\ M/N \arrow[r, "\overline{f}"'] & K \end{tikzcd}$$

Isomorphism theorems

The isomorphism theorems for abelian groups give isomorphism theorems for modules.

- If $f : M \to K$ is a homomorphism, then the map $\overline{f}$ gives an isomorphism between $M/\ker(f)$ and $f(M) \subset K$.
- $(M+N)/N$ is isomorphic to $M/(M \cap N)$.
- $(M/A)/(N/A)$ is isomorphic to M/N.
- There is a bijection between the lattice of submodules of M/N and submodules of M containing N given by $K \leftrightarrow K/N$.

The proofs of all of these facts are found by checking that the group isomorphisms respect the action of the ring R.

$\mathrm{Hom}_R(M, N)$

The set $\mathrm{Hom}_R(M, N)$ is an abelian group:
$(f + g)(m) = f(m) + g(m)$ and the zero map is the identity.

If R is commutative then $\mathrm{Hom}_R(M, N)$ is an R-module if we set (rf) to be the function $(rf)(m) = r(f(m)) = f(rm)$. We need rf to be a module homomorphism, which means we need:

$$(rf)(sm) = s(rf)(m).$$

This works out ok if R is commutative since

$$(rf)(sm) = f(rsm) = f(srm) = s(f(rm)) = s((rf)(m))$$

but it fails if R is not commutative.

$\mathrm{Hom}_R(M, M)$

The set $\mathrm{Hom}_R(M, M)$ is a ring with multiplication given by composition. The identity map gives an identity for this ring.

If R **is commutative** then, given $r \in R$, we have an element $\phi_r \in \mathrm{Hom}_R(M, M)$ given by $\phi_r(m) = rm$. This is a homomorphism because

$$\phi_r(sm) = rsm = srm = s\phi_r(m)$$

but this fails in general if R is not commutative. Thus, if R is commutative, $\mathrm{Hom}_R(M, M)$ is an R-algebra.

More on $\mathrm{Hom}_R(M, M)$

If $M = R^n$, then $\mathrm{Hom}_R(M, M)$ is the ring of $n \times n$ matrices with entries from R.