Potential Final Exam Questions (Unedited)

1. Let $S = \{\frac{1}{2^n} : n \in \mathbb{Z}\}$. Prove that S is countably infinite.

2. Let $A = \{0, 1, 2, 3, 4\}$ and $B = \{1, 3, 5\}$. Let f be the function

$$f \subset A \times B = \{(0,1), (3,3), (2,3), (1,1), (4,1)\}$$

and let $g : B \to A$ be the function

$$g \subset B \times A = \{(1,0), (3,2), (5,2)\}$$

1. What is $f(2)$?
2. Give $g \circ f$ as a subset of $A \times A$.

3. Prove that the set of positive real numbers $(0, \infty)$ has the same cardinality as the set $(-\infty, -3)$ of real numbers smaller than 3.

4. Prove that the set $\{-1, 0, 1\} \times \mathbb{N}$ is countable. You can give a proof by a diagram along with a *clear* explanation; you need not give a formula (although you may if you choose).

5. Let $A = \{1, 2, 3, 4, 5\}$.

1. Give an example of a function $f : A \to A$ that has $f(1) = 3$ and that is neither surjective nor injective.
2. Given an example of a function $g : A \to A$ that has $g(1) = 3$ and is bijective.

6. Let S be the set of infinite sequences $a_1, a_2, \ldots$ with all a_i either 0 or 1. Use Cantor's diagonalization argument to prove that S is uncountable.

7. Prove that, if $f : A \to B$ is bijective, and $g : B \to C$ is bijective, then $g \circ f : A \to C$ is bijective.

8. Let $\mathbb{Z}_{17}$ be the set of equivalence classes of integers modulo 17. Let $f : \mathbb{Z}_{17} \to \mathbb{Z}_{17}$ be the function $f(x) = 3x$. Find a so that $g(x) = ax$ is the inverse function to f.

9. Consider the function $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ given by the formula $f(x, y) = 7x + 11y$.

1. Prove that f is surjective.

2. Find (x, y) so that $f(x, y) = 4$.

10. Let r be a rational number less than 1. Prove that the sequence $a(n) = r^n$ for $n = 1, 2, \ldots$ converges to zero.

11. Express the repeating decimal $0.125125125\cdots$ as a geometric series and express the limit of that series as a fraction.

12. Prove that, if $f : A \to B$ has an inverse function $g : B \to A$, then f is injective.